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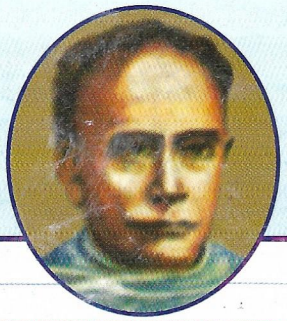

VIDYASAGAR UNIVERSITY

MIDNAPORE-721 102

M. Sc. in Mathematics

Part - II

Paper - IX

Module No. 101, 102, 103, 104, 105, 106, 107, 108

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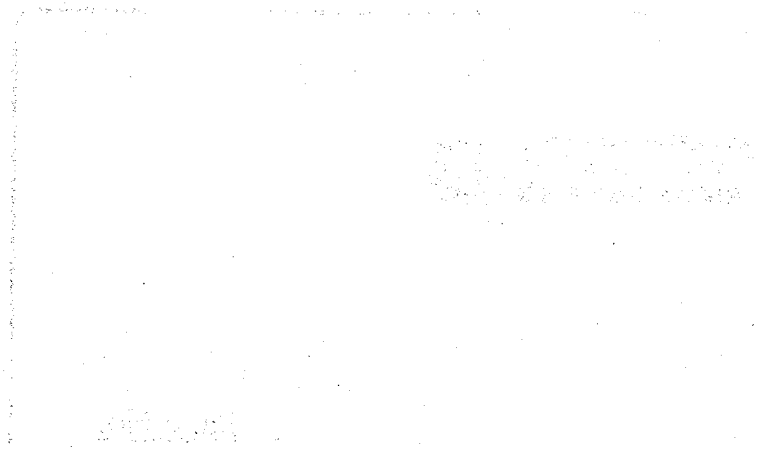

M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

Part -II
Paper - IX (Special Paper)
OPERATIONS RESEARCH
Module No. 101
(Revised Simplex Method)

REVISED SIMPLEX METHOD

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101.1 INTRODUCTION

The revised simplex method is used for reducing computational time of simplex method by arranging data in a smaller array. This method proceeds through the same steps as simplex method but keeps all important data in a smaller array. The 'revised' aspect concerns the procedure of changing the simplex tables only. The revised simplex method is thus an efficient computational procedure for solving a linear programming problem with less time and labour. For large size problem this method is found to be very useful as it reduces the cost of obtaining the solution.

101.2 OBJECTIVE

Linear programming is one of the most important optimization techniques developed in the field of

Operations Research. In Reality the LP problems are very large in size and so it becomes very tedious to get the solution by usual simplex method. To overcome this difficulty simplex method has been modified i.e. revised to tackel such large LP problems. Thus revised simplex method has become very important now-a-days. Specially to get the solution using computer it saves a lot of space and time. It is thus necessary to acquire knowledge of revised simplex method. This model will serve the purpose.

101.3 Revised Simplex Method :

When a linear programming problem is solved by simplex method, successive iterations are obtained by using suitable row operations so that the objective function reduces its value in each step if it is a problem of maximization. Also the net evaluations should remain always non-negative in every step. This method requires storing the entire table in the memory of the computer. For large size problem it may not be feasible. So, it requires to device a new method by modifing simplex method to handle LPP with large number of decision variables and constraints.

In fact, it is found that it is not necessary to compute the entire simplex table during each iteration. The only informations needed to pass from one table to the next one are seen to be

i) Net evaluations $z_j - c_j$ to determine the non-basic variable that enters the basis

ii) The key colounn

iii) The current basic variables and their values to determine the minimum positive ratio, and thereby to determine the basic variable that will leave the basis.

It is shown that all the above informations can be directly obtained from the original equations of the given LPP by making use of the inverse of the current basis matrix.

If B be the current basis then we have

$x_B = B^{-1}b$, $y_j = B^{-1}a_j$; for all $j = 1, 2, ..., n$

$z_j - c_j = c_B B^{-1} a_j - c_j$; for all $j = 1, 2, ..., n$

and $z = c_B x_B$

We note that all these necessary informations can be calculated if the current value of B^{-1} is known. Much computational work is needed for transformation of all y_j, $j = 1, 2, ..., n$. But all y_j are not needed to go to the next table. As noted above we need only to know the key column i.e., y_k. This will actually save our labour. At each iteration x_B, z, $c_B B^{-1}$ and B^{-1} are transformed and not all the y_j are transformed, only the key column y_k is transformed in the revised simplex method. The criteria for selecting the entering and departing vectors in the revised simplex method is precisely the same as that was in the simplex method. The labour saving point in this method lies in the fact of computing the inverse of the next basis directly from that of the current basis without actually having to invert the next basis.

101.4 Standard form for Revised Simplex Method

Let the linear programming problem be

Maximize $z = cx$

subject to $Ax = b$ ..... (101.4.1)

$x \geq 0$

where c, $x^T \in R^n$, $b^T \in R^m$ and A is an m×n real matrix. In the revised simplex method we consider the objective function equation $z = cx$ also as one constraint. Thus the new system becomes a (m+1) simultaneous linear equations in n+1 variables $z, x_1, x_2, ..., x_n$. The problem thus becomes to get the solution of this system such that z is as large as possible. The simultaneous linear system thus becomes

$Ax + oz = b$

$-cx + z = o$ ..... (101.4.2)

$x \geq 0$, z is unrestricted.

Hence the LPP (101.4.1) becomes equivalent to the problem of finding the solution of the system (101.4.2) such that z is as large as possible.

In matrix notation (101.4.2) becomes

$$\begin{bmatrix} A & 0 \\ -c & 1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}, \quad x \geq 0 \qquad \text{..... (101.4.3)}$$

Let B be the initial basis submatrix of A and $x_B = B^{-1}b$ be the initial basic feasible solution to the original LPP (101.4.1). Since the values of the non-basic variables are always zero (101.4.2) becomes

$Bx_B + oz = b$

$-c_B x_B + z = 0$ ..... (101.4.4)

or, $$\begin{bmatrix} B & 0 \\ -c_B & 1 \end{bmatrix} \begin{bmatrix} x_B \\ z \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

or, $\hat{B}\hat{x}_B = \hat{b}$ ..... (101.4.5)

where $\hat{B} = \begin{bmatrix} B & 0 \\ -c_B & 1 \end{bmatrix}$, $\hat{x}_B = \begin{bmatrix} x_B \\ z \end{bmatrix}$ and $\hat{b} = \begin{bmatrix} b \\ 0 \end{bmatrix}$ ..... (101.4.6)

From (1.4) we have

$\hat{x}_B = \hat{B}^{-1}\hat{b}$ ..... (101.4.7)

This is the initial basic feasible solution to the reformulated problem (101.4.2).

Computation of inverse of $\hat{B}$ by partitioning. :

We have $\hat{B} = \begin{bmatrix} B & 0 \\ -c_B & 1 \end{bmatrix}$

Let $\hat{B}^{-1} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ ..... (101.4.8)

Since, $\hat{B}\hat{B}^{-1} = I$ we have

$$\begin{bmatrix} B & 0 \\ -c_B & 1 \end{bmatrix}\begin{bmatrix} P & Q \\ R & S \end{bmatrix} = I_{m+1}$$

or, $\begin{bmatrix} BP + OR & BQ + OS \\ -c_BP + R & -c_BQ + S \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ O & 1 \end{bmatrix}$

$\therefore \quad BP = I_m$

$BQ = 0$

$-c_BP + R = 0$

$-c_BQ + S = 1$

Since B^{-1} exists, we get from above

$P = B^{-1}I_m = B^{-1}$

$Q = B^{-1}O = 0$

$R = c_BB^{-1}$

$S = 1 + c_BO = 1$

Thus from (101.4.8) we get

$$\hat{B}^{-1} = \begin{bmatrix} B^{-1} & 0 \\ c_BB^{-1} & 1 \end{bmatrix}$$ ..... (101.4.9)

We note that all the components of $\hat{B}^{-1}$ are known.

Determination of net evaluations, key column and BFS :

We define $\hat{A} = \begin{bmatrix} A \\ -c \end{bmatrix}$

and $\hat{y} = \hat{B}^{-1}\hat{A}$

Then $\hat{y} = \begin{bmatrix} B^{-1} & 0 \\ c_BB^{-1} & 1 \end{bmatrix}\begin{bmatrix} A \\ -c \end{bmatrix}$

$$= \begin{bmatrix} B^{-1}A & -Oc \\ c_BB^{-1}A - c \end{bmatrix}$$

$$= \begin{bmatrix} B^{-1}A \\ c_B(B^{-1}A) - c \end{bmatrix}$$ ..... (101.4.10)

We have

$A = By$

$\therefore \quad y = B^{-1}A$

$\therefore$ From (101.4.10) we have $\hat{y} = \begin{bmatrix} y \\ c_B y - c \end{bmatrix}$

or, $\begin{bmatrix} \hat{y}_1 & \hat{y}_2 & \cdots & \hat{y}_n \end{bmatrix} = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ z_1 - c_1 & z_2 - c_2 & \cdots & z_n - c_n \end{bmatrix}$

Thus for $j = 1, 2, \ldots, n$ we have

$\hat{y}_j = \begin{bmatrix} y_j \\ z_j - c_j \end{bmatrix}$ and $y_j = B^{-1}a_j$.

Hence the net evaluations are the components of $c_B B^{-1}A - c$

i.e., $c_B B^{-1}A - c = [z_1 - c_1 \quad z_2 - c_2 \quad \ldots \quad z_n - c_n]$

Most negative $z_j - c_j$ will determine the key column. let $z_k - c_k$ be the most negative $z_j - c_j$. Then the key column is

$$\hat{y}_k = \begin{bmatrix} y_k \\ z_k - c_k \end{bmatrix} = \begin{bmatrix} B^{-1}a_k \\ z_k - c_k \end{bmatrix} \qquad \ldots\ldots\ldots\ldots (101.4.11)$$

From (101.4.7) and (101.4.6) we have

$$\hat{x}_B = \begin{bmatrix} x_B \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b} = \begin{bmatrix} B^{-1} & 0 \\ c_B B^{-1} & 1 \end{bmatrix}\begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} B^{-1}b \\ c_B B^{-1}b \end{bmatrix} = \begin{bmatrix} B^{-1}b \\ c_B x_B \end{bmatrix} \qquad \ldots\ldots\ldots\ldots (101.4.12)$$

We note the important fact that all necessary informations can be obtained from the products $\hat{B}^{-1}\hat{A}$ and $\hat{B}^{-1}\hat{b}$. Also we note that $\hat{A}$ and $\hat{b}$ remains same in all steps, only $\hat{B}^{-1}$ changes in each step of simplex table depending on the current basis B.

The above discussion enables us to state the stepwise revised simplex method.

101.5. Computational Procedure of Revised Simplex Method :

The iteration procedure of revised simplex method are as follows.

Step 1. Introduce necessary slack and surplus variables. Convert the problem into a problem of maximization if it is in minimization form. Restate the LPP in the standard form of revised simplex method i.e. in the form

$$\begin{bmatrix} A & 0 \\ -c & 1 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix}=\begin{bmatrix} b \\ 0 \end{bmatrix}, \quad x \geq 0, \quad z \text{ is unrestricted.}$$

Step 2. Begin with the initial basis $B = I_m$ and form the auxiliary matrix $\hat{B} = \begin{bmatrix} B & 0 \\ -c_B & 1 \end{bmatrix}$ and write down

$\hat{B}^{-1} = \begin{bmatrix} B & 0 \\ c_B B^{-1} & 1 \end{bmatrix}$. Form $A = \begin{bmatrix} A \\ -c \end{bmatrix}$ and $\hat{b} = \begin{bmatrix} b \\ 0 \end{bmatrix}$,

Also form $\hat{x}_B = \begin{bmatrix} x_B \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b}$

Step 3. Compute the net evaluations $z_1 - c_1, z_2 - c_2, \ldots, z_n - c_n$

as the components of the product

$$\begin{bmatrix} c_B B^{-1} & 1 \end{bmatrix}\begin{bmatrix} A \\ -c \end{bmatrix}$$

If all $z_j - c_j$ are non-negative, the current basic solution $\hat{x}_B = \begin{bmatrix} x_B \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b}$ gives the optimal BFS and maximum value of the objective function.

If at least one $z_j - c_j$ is negative, determine the most negative of them. If $z_k - c_k$ is the most negative $z_j - c_j$ then find $\hat{y}_k = \begin{bmatrix} y_k \\ z_k - c_k \end{bmatrix} = \hat{B}^{-1}\hat{a}_k$. Go to step 4. If there is a tie for the most negative $z_j - c_j$, resolve the tie by any standard method. Take x_k as the new basic variable. Go to step 4.

Step 4. If all $y_{ik} \leq 0$ then there exists an unbounded solution to the given problem.

If at least one $y_{ik} > 0$, consider the current x_B and compute the replacement ratios

$$\left\{\frac{x_{B_i}}{y_{ik}} : y_{ik} > 0\right\}$$

If $\frac{x_{B_r}}{y_{rk}}$ is the minimum of all these ratios then the basic variable x_{B_r} becomes non-basic variable in the next table. i.e. x_{B_r} is replaced by x_k. Go to step 5.

Step 5. Write down the results obtained in steps 2, 3 and 4 in a table. This table is known as revised simplex table. This table is of the form

$\hat{y}_B$	$\hat{x}_B$	$\hat{B}^{-1}$	$\hat{y}_k$	$\frac{x_{Bi}}{y_{ik}} : y_{ik} > 0$

Step 6. Convert the key element y_{rk} of $\hat{y}_k$ into unity and all other elements into zero by suitable row operations. Same operations are to be applied in the current $\hat{B}^{-1}$. These operations will change $\hat{B}^{-1}$ to new $\hat{B}^{-1}$ for the next table.

Step 7. Consider new $\hat{B}^{-1}$ obtained in step 6 as $\hat{B}^{-1}$ and go to step 3. Repeat the procedure until an optimum basic feasible solution is obtained or there is an indication of an unbounded solution.

101.6 Advantages of revised simplex method

The advantages of the revised simplex method over the regular simplex method are

i) fewer calculations are required

ii) less storage is needed when computing the problem on a computer

iii) the round off errors can be controlled as table entries are not repeatedly recalculated.

101.7 Comparison of Simplex Method and Revised Simplex Method

Let us consider the LPP

Maximize $z = cx$

subject to $Ax = b, x \geq 0$

where A is a matrix of order $m \times n$. If initially artificial variables are not needed for obtaining the initial basis matrix, then for solving this problem by the simplex method we have to transfer $(n+1)$ columns at each iteration (n columns for A and one column for x_B). Also, at each iteration one variable is introduced into the basis and one is removed from it. Thus, in total we compute for $(n-m+1)$ columns. Furthermore, for each of these columns, we have to transform $(m+1)$ elements. For moving from one iteration to another we also need to calculate minimum

ratio x_B/y_k. Hence in all we have to perform multiplication $(m+1)(n-m+1)$ times and addition $m(n-m+1)$ times.

In the revised simplex method, there are (m+1) rows and (m+2) columns. So, for moving from one iteration to another we have to make $(m+1)^2$ multiplication operations to get an improved solution in addition to m(n–m) operations for calculating (z_j-c_j)'s.

In the revised simplex method we need to make (m+1) (m+2) entries in each table while in simplex method there are (m+1) (n+1) entries in each table.

If the number of variables n is significantly larger than the number of constraints m, then the computational efforts of the revised simplex method is smaller than that of the simplex method.

Revised simplex method reduces the cumulative round-off error while calculating (z_j-c_j)'s and updated column y_k due to the use of original data.

The inverse of the current basis matrix is obtained automatically.

101.8. Illustrative Examples

Example 101.8.1. Use revised simplex method to solve the LPP

Maximize $z = x_1 + x_2$

subject to $3x_1 + 2x_2 \le 6$

$x_1 + 4x_2 \le 4$

$x_1, x_2 \ge 0$

Solution : After the introduction of slack variables $x_3 \ge 0$ and $x_4 \ge 0$, the given LPP is re-written in standard form as to

Maximize $z = cx$

subject to $Ax = b$ and $x \ge 0$

where $A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{bmatrix}$, $C = [1100]$, $b = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$\therefore$ We have $\hat{A} = \begin{bmatrix} A \\ -c \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 1 & 4 & 0 & 1 \\ -1 & -1 & 0 & 0 \end{bmatrix}$, $\hat{b} = \begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $x_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$, $c_B = [c_3 \quad c_4] = [0 \quad 0]$

Now $c_B B^{-1} = [0 \quad 0]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [0 \quad 0]$

$$\therefore \hat{B}^{-1} = \begin{bmatrix} B^{-1} & 0 \\ c_B B^{-1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and so $\hat{x}_B = \begin{bmatrix} x_B \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix} \quad \therefore x_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ and $z = 0$.

The net evaluations are computed as follows

$$\left[c_B B^{-1}\right]\begin{bmatrix} A \\ -c \end{bmatrix} = [0 \quad 0 \quad 1]\begin{bmatrix} 3 & 2 & 1 & 0 \\ 1 & 4 & 0 & 1 \\ -1 & -1 & 0 & 0 \end{bmatrix} = [-1 \quad -1 \quad 0 \quad 0]$$

$$= [z_1 - c_1 \quad z_2 - c_2 \quad z_3 - c_3 \quad z_4 - c_4]$$

Since there are negative netevalutions, the solution obtained is not optimal. The most negative net evalutions are $z_1 - c_1 = -1, z_2 - c_2 = -1$, There being a tie, we arbitrarily choose $z_1 - c_1$. Therefore x_1 is the new incoming basic variable.

Now we compute

$$\hat{y}_1 = \hat{B}^{-1}\hat{a}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

These results are shown in the following initial revised simplex table

Basic variables	values	$\hat{B}^{-1}$			$\hat{y}_1$	min ratio
x_3	6	1	0	0	[3]	2 →
x_4	4	0	1	0	1	4
z	0	0	0	1	−1	

Here the minimum ratio is Min $\left\{\frac{x_{Bi}}{y_{ik}} : y_{ik>0}\right\} = 2$ and the corresponding variable is x_3. Therefore, the outgoing basic variable is x_3. So x_3 is replaced by x_1 in the next table.

Using elementary row operations $\hat{y}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$ is converted to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Same operations is carried out in $\hat{B}^{-1}$.

This gives new $\hat{B}^{-1}$ as follows

$$\text{New } \hat{B}^{-1} = \begin{bmatrix} 1/3 & 0 & 0 \\ -1/3 & 1 & 0 \\ 1/3 & 0 & 1 \end{bmatrix}$$

The next BFS is given by

$$\hat{x}_B = \begin{bmatrix} x_1 \\ x_4 \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b} = \begin{bmatrix} 1/3 & 0 & 0 \\ -1/3 & 1 & 0 \\ 1/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \text{ and } z = 2.$$

The net evaluations are given by

$$\begin{bmatrix} c_B B^{-1} & 1 \end{bmatrix} \begin{bmatrix} A \\ -c \end{bmatrix} = \begin{bmatrix} 1/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 & 0 \\ 1 & 4 & 0 & 1 \\ -1 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1/3 & 1/3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} z_1 - c_1 & z_2 - c_2 & z_3 - c_3 & z_4 - c_4 \end{bmatrix}$$

Since there is negative net evaluations, the BFS is not optimal. Here the most negative (only negative) net evaluation is $z_2 - c_2 = -1/3$. So x_2 is the next incoming basic variable.

Now we compute

$$\hat{y}_2 = \hat{B}^{-1}\hat{a}_2 = \begin{bmatrix} 1/3 & 0 & 0 \\ -1/3 & 1 & 0 \\ 1/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 10/3 \\ -1/3 \end{bmatrix}$$

These results are shown in the following revised simplex table.

Basic variables	values	$\hat{B}^{-1}$			$\hat{y}_2$	min ratio
x_1	2	1/3	0	1	2/3	3
x_4	2	-1/3	1	0	10/3	3/5 →
z	2	1/3	0	1	−1/3	

Here the minimum ratio is 3/5 and is associated with the basic variable x_4. Therefore, the outgoing basic variable is x_4. So x_4 is replaced by x_2 in the next table. Using elementary row operations $\hat{y}_2 = \begin{bmatrix} 2/3 \\ 10/3 \\ -1/3 \end{bmatrix}$ is converted to $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and the same operations are performed in $\hat{B}^{-1}$. This gives new $\hat{B}^{-1}$ as follows

$$\text{New } \hat{B}^{-1} = \begin{bmatrix} 2/5 & -1/5 & 1 \\ -1/10 & 3/10 & 0 \\ 3/10 & 1/10 & 1 \end{bmatrix}$$

The next BFS is given by

$$\hat{x}_B = \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b} = \begin{bmatrix} 2/5 & -1/5 & 1 \\ -1/10 & 3/10 & 0 \\ 3/10 & 1/10 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 8/5 \\ 3/5 \\ 11/5 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8/5 \\ 3/5 \end{bmatrix} \text{ and } z = 11/5$$

The net evaluations are given by

$$\begin{bmatrix} c_B B^{-1} & 1 \end{bmatrix}\begin{bmatrix} A \\ -c \end{bmatrix} = \begin{bmatrix} \frac{3}{10} & \frac{1}{10} & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 & 0 \\ 1 & 4 & 0 & 1 \\ -1 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{3}{10} & \frac{1}{10} \end{bmatrix}$$

Here all net evaluations are non-negative. Hence we have obtained the optimal solution and the optimal solution is $x_1 = 8/5$, $x_2 = 3/5$and $z_{max} = 11/5$.

Example 101.8.2. Use revised simplex method to solve the LPP

Maximize $z = 2x_1 - 3x_2 + x_3$

subject to $3x_1 + 6x_2 + x_3 \le 6$

$4x_1 + 2x_2 + x_3 \le 4$

$x_1 - x_2 + x_3 \le 3$

$x_1, x_2, x_3 \ge 0$

Solution : Introducing slack variables $x_4 \ge 0, x_5 \ge 0, x_6 \ge 0$, the given LPP becomes in standard form as

Maximize $z = 2x_1 - 3x_2 + x_3 + 0x_4 + 0x_5 + 0x_6$

subject to
$$3x_1 + 6x_2 + x_3 + x_4 = 6$$
$$4x_1 + 2x_2 + x_3 + x_5 = 4$$
$$x_1 - x_2 + x_3 + x_6 = 3$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

or, Maximize $z = cx$

subject to $Ax = b, x \geq 0$

where $A = \begin{bmatrix} 3 & 6 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{bmatrix}$, $c = [2 \quad -3 \quad 1 \quad 0 \quad 0 \quad 0]$

$b = \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix}$ and $x = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6]^T$.

We have $\hat{A} = \begin{bmatrix} A \\ -c \end{bmatrix} = \begin{bmatrix} 3 & 6 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \\ -2 & 3 & -1 & 0 & 0 & 0 \end{bmatrix}$, $\hat{b} = \begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 3 \\ 0 \end{bmatrix}$

Initially $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $x_B = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix}$, $c_B = [c_4 \quad c_5 \quad c_6] = [0 \quad 0 \quad 0]$

Now $c_B B^{-1} = [0 \quad 0 \quad 0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [0 \quad 0 \quad 0]$

$$\therefore \hat{B}^{-1} = \begin{bmatrix} B^{-1} & 0 \\ c_B B^{-1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \hat{x}_B = \begin{bmatrix} x_B \\ z \end{bmatrix} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 3 \\ 0 \end{bmatrix}$$

i.e. $\begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix}$ and $z = 0$.

The net evaluations are the components of

$$\begin{bmatrix} c_B B^{-1} & 1 \end{bmatrix} \begin{bmatrix} A \\ -c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 6 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \\ -2 & 3 & -1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 3 & -1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} z_1 - c_1 & z_2 - c_2 & z_3 - c_3 & z_4 - c_4 & z_5 - c_5 & z_6 - c_6 \end{bmatrix}$$

Since there are negative net evaluations, the solution obtained is not optimal. The most negative net evaluation is $z_1 - c_1 = -2$. Therefore x_1 will be the new basic variable.

Now we compute

$$\hat{y}_1 = \hat{B}^{-1}\hat{a}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \\ -2 \end{bmatrix}$$

These results are shown in the following initial revised simplex table

Basic variables	values	$\hat{B}^{-1}$				$\hat{y}_1$	min ratio
x_4	6	1	0	0	0	3	2
x_5	4	0	1	0	0	4	1 →
x_6	3	0	0	1	0	1	3
z	0	0	0	0	1	-2	

Here the minimum ratio in Min $\left\{ \frac{x_{Bi}}{y_{ik}} : y_{ik} > 0 \right\} = 1$ and the corresponding variable is x_5. Therefore, the outgoing basic variable is x_5. So x_5 is replaced by x_1 in the next table.

Using elementary row operations $\hat{y}_1 = \begin{bmatrix} 3 \\ 4 \\ 1 \\ -2 \end{bmatrix}$ is converted to $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and the same operations are done for $\hat{B}^{-1}$. This gives new $\hat{B}^{-1}$ as follows

$$\hat{B}^{-1} = \begin{bmatrix} 1 & -3/4 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & -1/4 & 1 & 0 \\ 0 & 1/2 & 0 & 1 \end{bmatrix}$$

The new BFS is given by

$$\hat{x}_B = \begin{bmatrix} x_4 \\ x_1 \\ x_6 \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b} = \begin{bmatrix} 1 & -3/4 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & -1/4 & 1 & 0 \\ 0 & 1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\therefore \quad x_B = \begin{bmatrix} x_4 \\ x_1 \\ x_6 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \text{ and } z = 2.$$

The net evaluations are given by

$$\begin{bmatrix} c_B B^{-1} & 1 \end{bmatrix} \begin{bmatrix} A \\ -c \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 6 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \\ -2 & 3 & -1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 & -1/2 & 0 & 1/2 & 0 \end{bmatrix}$$

Since there is negative net evaluation, the BFS obtained is not optimal. Here the only negative net evaluation is $z_3 - c_3 = -1/2$. So x_3 is the next incoming basic variable.

Now we compute

$$\hat{y}_3 = \hat{B}^{-1}\hat{a}_3 = \begin{bmatrix} 1 & -3/4 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & -1/4 & 1 & 0 \\ 0 & 1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/4 \\ 3/4 \\ -1/2 \end{bmatrix}$$

These results are shown in the following revised simplex table

Basic variables	values	$\hat{B}^{-1}$				$\hat{y}_3$	min ratio
x_4	3	1	-3/4	0	0	1/4	12
x_1	1	0	1/4	0	0	1/4	4
x_6	2	0	-1/4	1	0	3/4	8/3 →
z	2	0	1/2	0	1	-1/2	

Here the minimum ratio is 8/3 and is associated with the basic variable x_6. Therefore, the outgoing basic variable is x_6. So x_6 is replaced by x_3 in the next iteration. Using elementary row operations $\hat{y}_3 = \begin{bmatrix} 1/4 \\ 1/4 \\ 3/4 \\ -1/2 \end{bmatrix}$ is converted to $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and the same operations are performed in $\hat{B}^{-1}$. This gives the new $\hat{B}^{-1}$ as follows

$$\text{New } \hat{B}^{-1} = \begin{bmatrix} 1 & -2/3 & -1/3 & 0 \\ 0 & 1/3 & -1/3 & 0 \\ 0 & -1/3 & 4/3 & 0 \\ 0 & 1/3 & 2/3 & 1 \end{bmatrix}$$

The next BFS is given by

$$\hat{x}_B = \begin{bmatrix} x_4 \\ x_1 \\ x_3 \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b} = \begin{bmatrix} 1 & -2/3 & -1/3 & 0 \\ 0 & 1/3 & -1/3 & 0 \\ 0 & -1/3 & 4/3 & 0 \\ 0 & 1/3 & 2/3 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 1/3 \\ 8/3 \\ 10/3 \end{bmatrix}$$

$$\therefore \quad x_B = \begin{bmatrix} x_4 \\ x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 1/3 \\ 8/3 \end{bmatrix} \text{ and } z = 10/3.$$

The net evaluations are given by

$$\left[c_B B^{-1} \quad 1\right]\begin{bmatrix} A \\ -c \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 2/3 & 1 \end{bmatrix}\begin{bmatrix} 3 & 6 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \\ -2 & 3 & -1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 & 0 & 1/3 & 2/3 \end{bmatrix}$$

Here all net evaluations are non-negative. Hence we have obtained the optimal solution. The optimal solution is

$x_1 = 1/3$, $x_2 = 0$, $x_3 = 8/3$ and $z = 10/3$.

Example 101.8.3. Use revised simplex method to solve the LPP

Maximize $z = x_1 + 2x_2$

subject to $2x_1 + 5x_2 \geq 6$

$x_1 + x_2 \geq 2$

$x_1, x_2 \geq 0$

Solution : Introducing surplus variables $x_3 \geq 0$, $x_4 \geq 0$ and artificial variables $x_5 \geq 0$, $x_6 \geq 0$, the given LPP becomes in standard form as

Maximize $z = x_1 + 2x_2 - x_3 - x_4 - Mx_5 - Mx_6$

subject to $2x_1 + 5x_2 - x_3 + x_5 = 6$

$x_1 + x_2 - x_4 + x_6 = 2$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

or, Maximize $z = cx$

subject to $Ax = b, x \geq 0$

where $A = \begin{bmatrix} 2 & 5 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 & 1 \end{bmatrix}$, $c = \begin{bmatrix} 1 & 2 & -1 & -1 & -M & -M \end{bmatrix}$

$x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{bmatrix}^T$ and $b = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

$\therefore$ We have $\hat{A} = \begin{bmatrix} A \\ -c \end{bmatrix} = \begin{bmatrix} 2 & 5 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 & 1 \\ -1 & -2 & 1 & 1 & M & M \end{bmatrix}$, $\hat{b} = \begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $x_B = \begin{bmatrix} x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$, $c_B = \begin{bmatrix} c_5 & c_6 \end{bmatrix} = \begin{bmatrix} -M & -M \end{bmatrix}$

Now $c_B B^{-1} = [-M \quad -M]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [-M \quad -M]$

$$\therefore \hat{B}^{-1} = \begin{bmatrix} B^{-1} & 0 \\ c_B B^{-1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -M & -M & 1 \end{bmatrix}$$

$$\therefore \hat{x}_B = \begin{bmatrix} x_B \\ z \end{bmatrix} = \begin{bmatrix} x_5 \\ x_6 \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -M & -M & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ -8M \end{bmatrix}$$

$$\therefore x_B = \begin{bmatrix} x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \text{ and } z = -8M.$$

The net evaluations are the components of

$$\begin{bmatrix} c_B B^{-1} & 1 \end{bmatrix}\begin{bmatrix} A \\ -c \end{bmatrix} = [-M \quad -M \quad 1]\begin{bmatrix} 2 & 5 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 & 1 \\ -1 & -2 & 0 & 0 & M & M \end{bmatrix} = [-3M-1 \quad -6M-2 \quad M \quad M \quad 0 \quad 0]$$

Since there are negative net evaluations, the solution obtained is not optimal. The most negative net evaluation is $z_2 - c_2 = -6M - 2$. Therefore x_2 will be the new basic variable.

New we compute

$$\hat{y}_2 = \hat{B}^{-1}\hat{a}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -M & -M & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -6M-2 \end{bmatrix}$$

These results are shown in the following initial revised simplex table

Basic variables	values	$\hat{B}^{-1}$			$\hat{y}_2$	min ratio
x_5	6	1	0	0	[5]	6/5 →
x_6	2	0	1	0	1	2
z	-8M	-M	-M	1	-6M-2	

Here the minimum ratio is $\text{Min}\left\{\frac{x_{Bi}}{y_{ik}} : y_{ik} > 0\right\} = 6/5$ and the corresponding variable is x_5. Therefore the outgoing basic variable is x_5. So x_5 is replaced by x_2 in the next iteration.

Using elementary row operations $\hat{y}_2 = \begin{bmatrix} 5 \\ 1 \\ -6M-2 \end{bmatrix}$ is converted to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and the same operations are done for $\hat{B}^{-1}$. This gives new $\hat{B}^{-1}$ as follows

$$\text{New } \hat{B}^{-1} = \begin{bmatrix} 1/5 & 0 & 0 \\ -1/5 & 1 & 0 \\ \frac{M+2}{5} & -M & 1 \end{bmatrix}$$

The new BFS is given by

$$\hat{x}_B = \begin{bmatrix} x_B \\ z \end{bmatrix} = \begin{bmatrix} x_2 \\ x_6 \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b} = \begin{bmatrix} 1/5 & 0 & 0 \\ -1/5 & 1 & 0 \\ \frac{M+2}{5} & -M & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6/5 \\ 4/5 \\ \frac{-4M+12}{5} \end{bmatrix}$$

$$\therefore \quad x_B = \begin{bmatrix} x_2 \\ x_6 \end{bmatrix} = \begin{bmatrix} 6/5 \\ 4/5 \end{bmatrix} \text{ and } z = \frac{-4M+12}{5}$$

The net evaluations are given by

$$\begin{bmatrix} c_B B^{-1} & 1 \end{bmatrix} \begin{bmatrix} A \\ -c \end{bmatrix} = \begin{bmatrix} \frac{M+2}{5} & -M & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 & 1 \\ -1 & -2 & 0 & 0 & M & M \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-3M-1}{5} & 0 & \frac{-M-2}{5} & M & \frac{6M+2}{5} & 0 \end{bmatrix}$$

Since there are negative net evaluations, the BFS obtained is not optimal. The most negative net evaluation is $z_1 - c_1 = \frac{-3M-1}{5}$. So x_1 is the next incoming basic variable.

Now we compute

$$\hat{y}_1 = \hat{B}^{-1}\hat{a}_1 = \begin{bmatrix} 1/5 & 0 & 0 \\ -1/5 & 1 & 0 \\ \frac{M+2}{5} & -M & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 3/5 \\ \frac{-3M-1}{5} \end{bmatrix}$$

These results are shown in the following revised simplex table

Basic variables	values	$\hat{B}^{-1}$			$\hat{y}_1$	min ratio
x_2	6/5	1/5	0	0	2/5	3
x_6	4/5	−1/5	1	0	3/5	4/3 →
z	$\frac{-4M+12}{5}$	$\frac{M+2}{5}$	−M	1	$\frac{-3M-1}{5}$	

Here the minimum ratio is 4/3 and is associated with the basic variable x_6. So x_6 is replaced by x_1 in the next iteration. Using elementary row operations $\hat{y}_1 = \left[2/5 \quad 3/5 \quad \frac{-3M-1}{5}\right]^T$ is converted to $[0 \quad 1 \quad 0]^T$ and the same operations are performed in $\hat{B}^{-1}$. This gives the new $\hat{B}^{-1}$ as follows

$$\text{New } \hat{B}^{-1} = \begin{bmatrix} 1/3 & -2/3 & 0 \\ -1/3 & 5/3 & 0 \\ 1/3 & 1/3 & 1 \end{bmatrix}$$

The next BFS is given by

$$\hat{x}_B = \begin{bmatrix} x_B \\ z \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b} = \begin{bmatrix} 1/3 & -2/3 & 0 \\ -1/3 & 5/3 & 0 \\ 1/3 & 1/3 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 4/3 \\ 8/3 \end{bmatrix}$$

$$\therefore \quad x_B = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 4/3 \end{bmatrix} \text{ and } z = 8/3.$$

The net evaluations are given by

$$\begin{bmatrix} c_B B^{-1} & 1 \end{bmatrix}\begin{bmatrix} A \\ -c \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 5 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 & 1 \\ -1 & -2 & 0 & 0 & M & M \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -1/3 & -1/3 & \frac{1}{3}+M & \frac{1}{3}+M \end{bmatrix}$$

Since there are negative net evaluations, the BFS obtained is not optimal. The most negative net evaluations are $z_3 - c_3$ and $z_4 - c_4$. We choose $z_3 - c_3$ arbitrarily. So x_3 is the incoming basic variable.

Now we compute

$$\hat{y}_3 = \hat{B}^{-1}\hat{a}_3 = \begin{bmatrix} 1/3 & -2/3 & 0 \\ -1/3 & 5/3 & 0 \\ 1/3 & 1/3 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 1/3 \\ -1/3 \end{bmatrix}$$

These results are shown in the following revised simplex table

Basic variables	values	$\hat{B}^{-1}$			$\hat{y}_3$	min ratio
x_2	2/3	1/3	−2/3	0	−1/3	....
x_1	4/3	−1/3	5/3	0	1/3	4 →
z	8/3	1/3	1/3	1	−1/3	

Here the minimum ratio is 4 and is associated with the basic variable x_1. So x_1 is replaced by x_3 in the next iteration. Using elementary row operations $\hat{y}_3 = \begin{bmatrix} -1/3 \\ 1/3 \\ -1/3 \end{bmatrix}$ is converted to $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and the same operations are performed in $\hat{B}^{-1}$. This gives the next $\hat{B}^{-1}$ as follows.

$$\text{New } \hat{B}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 5 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

The next BFS is given by

$$\hat{x}_B = \begin{bmatrix} x_B \\ z \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 5 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

$$\therefore\ x_B = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \text{ and } z = 4.$$

The net evaluations are given by

$$\begin{bmatrix} c_B B^{-1} & 1 \end{bmatrix}\begin{bmatrix} A \\ -c \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 5 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 & 1 \\ -1 & -2 & 0 & 0 & M & M \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 & M & M+2 \end{bmatrix}$$

Here $z_4 - c_4 = -2$ is the only negative net evaluation, so the BFS obtained is not optimal. The next incoming basic variable is x_4.

Now we compute

$$\hat{y}_4 = \hat{B}^{-1}\hat{a}_4 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 5 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ -2 \end{bmatrix}$$

Here, since all the entries of the column $\hat{y}_4$ are negative, the LPP has unbounded solution.

Example 101.8.4. : Solve by revised simplex method

Maximize $z = 5x_1 + 3x_2$

subject to $4x_1 + 5x_2 \geq 10$

$5x_1 + 2x_2 \leq 10$

$3x_1 + 8x_2 \leq 12$

$x_1, x_2 \geq 0$

Solution : Introducing surplus variable $x_3 \geq 0$, slack variables $x_4 \geq 0$, $x_5 \geq 0$ and artificial variable $x_6 \geq 0$ the standard form of the given LPP is

Maximize $z = 5x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5 - Mx_6$

subject to $4x_1 + 5x_2 - x_3 + x_6 = 10$

$5x_1 + 2x_2 + x_4 = 10$

$3x_1 + 8x_2 + x_5 = 12$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

or, Maximize $z = cx$

subject to $Ax = b$, $x \geq 0$

where $$A = \begin{bmatrix} 4 & 5 & -1 & 0 & 0 & 1 \\ 5 & 2 & 0 & 1 & 0 & 0 \\ 3 & 8 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 5 & 3 & 0 & 0 & 0 & -M \end{bmatrix}$$

$$b = \begin{bmatrix} 10 \\ 10 \\ 12 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{bmatrix}^T.$$

$\therefore$ We have $$\hat{A} = \begin{bmatrix} A \\ -c \end{bmatrix} = \begin{bmatrix} 4 & 5 & -1 & 0 & 0 & 1 \\ 5 & 2 & 0 & 1 & 0 & 0 \\ 3 & 8 & 0 & 0 & 1 & 0 \\ -5 & -3 & 0 & 0 & 0 & M \end{bmatrix}, \quad \hat{b} = \begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 12 \\ 0 \end{bmatrix}$$

Initially, $B=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, $x_B=\begin{bmatrix}x_6\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}10\\10\\12\end{bmatrix}$, $c_B=[c_6 \quad c_4 \quad c_5]=[-M \quad 0 \quad 0]$

$$\therefore B^{-1}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

$$\text{Now } c_BB^{-1}=[-M \quad 0 \quad 0]\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=[-M \quad 0 \quad 0]$$

$$\therefore \hat{B}^{-1}=\begin{bmatrix}B^{-1}&0\\c_BB^{-1}&1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\-M&0&0&1\end{bmatrix}$$

$$\therefore \hat{x}_B=\begin{bmatrix}x_B\\z\end{bmatrix}=\begin{bmatrix}x_6\\x_4\\x_5\\z\end{bmatrix}=\hat{B}^{-1}\hat{b}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\-M&0&0&1\end{bmatrix}\begin{bmatrix}10\\10\\12\\0\end{bmatrix}=\begin{bmatrix}10\\10\\12\\-10M\end{bmatrix}$$

$$\therefore \begin{bmatrix}x_6\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}10\\10\\12\end{bmatrix} \text{ and } z=-10M.$$

The net evaluations are the components of

$$\begin{bmatrix}c_BB^{-1} & 1\end{bmatrix}\begin{bmatrix}A\\-c\end{bmatrix}=[-M \quad 0 \quad 0 \quad 1]\begin{bmatrix}4&5&-1&0&0&1\\5&2&0&1&0&0\\3&8&0&0&1&0\\-5&-3&0&0&0&M\end{bmatrix}$$

$$=[-4M-5 \quad -5M-3 \quad M \quad 0 \quad 0 \quad 0]$$

$$=[z_1-c_1 \quad z_2-c_2 \quad z_3-c_3 \quad z_4-c_4 \quad z_5-c_5 \quad z_6-c_6]$$

Since there are negative net evaluations, the solution obtained is not optimal. The most negative net evaluation is $z_2-c_2=-5M-3$. Therefore x_2 will be the new basic variable.

Now we compute $\hat{y}_2 = \hat{B}^{-1}\hat{a}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -M & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 8 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 8 \\ -5M-3 \end{bmatrix}$

These results are shown in the following initial revised simplex table

Basic variables	values	$\hat{B}^{-1}$				$\hat{y}_2$	min ratio
x_6	10	1	0	0	0	5	2
x_4	10	0	1	0	0	2	5
x_5	12	0	0	1	0	8	3/2 →
z	−10M	−M	0	0	1	−5M−3	

Here the minimum ratio is $\min\left\{\frac{x_{Bi}}{y_{ik}} : y_{ik} > 0\right\} = 3/2$ and the corresponding variable is x_5. Therefore, the outgoing basic variable is x_5. So x_5 is replaced by x_2 in the next table.

Using elementary row operations $\hat{y}_2 = \begin{bmatrix} 5 \\ 2 \\ 8 \\ -5M-3 \end{bmatrix}$ is converted to $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and the same operations are done for $\hat{B}^{-1}$. This gives new $\hat{B}^{-1}$ as follows

$$\hat{B}^{-1} = \begin{bmatrix} 1 & 0 & -5/8 & 0 \\ 0 & 1 & -1/4 & 0 \\ 0 & 0 & 1/8 & 0 \\ -M & 0 & \frac{5M+3}{8} & 1 \end{bmatrix}$$

The new BFS is given by

$$\hat{x}_B = \begin{bmatrix} x_6 \\ x_4 \\ x_2 \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b} = \begin{bmatrix} 1 & 0 & -5/8 & 0 \\ 0 & 1 & -1/4 & 0 \\ 0 & 0 & 1/8 & 0 \\ -M & 0 & \frac{5M+3}{8} & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 12 \\ 0 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 7 \\ 3/2 \\ \frac{-5M+9}{8} \end{bmatrix}$$

$$\therefore\ x_B = \begin{bmatrix} x_6 \\ x_4 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 7 \\ 3/2 \end{bmatrix} \text{ and } z = \frac{-5M+9}{8}$$

The net evaluations are the components of

$$\begin{bmatrix} c_B B^{-1} & 1 \end{bmatrix} \begin{bmatrix} A \\ -c \end{bmatrix} = \begin{bmatrix} -M & 0 & \frac{5M+3}{8} & 1 \end{bmatrix} \begin{bmatrix} 4 & 5 & -1 & 0 & 0 & 1 \\ 5 & 2 & 0 & 1 & 0 & 0 \\ 3 & 8 & 0 & 0 & 1 & 0 \\ -5 & -3 & 0 & 0 & 0 & M \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-17M-31}{8} & 0 & M & 0 & \frac{5M+3}{8} & 0 \end{bmatrix}$$

Since there is negative net evaluation, the BFS obtained is not optimal. Here the only negative net evaluation is $z_1 - c_1$. So x_1 is the next incoming basic variable.

Now we compute

$$\hat{y}_1 = \hat{B}^{-1}\hat{a}_1 = \begin{bmatrix} 1 & 0 & -5/8 & 0 \\ 0 & 1 & -1/4 & 0 \\ 0 & 0 & 1/8 & 0 \\ -M & 0 & \frac{5M+3}{8} & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 17/8 \\ 17/4 \\ 3/8 \\ \frac{-17M-31}{8} \end{bmatrix}$$

These results are shown in the following revised simplex table

Basic variables	values	$\hat{B}^{-1}$				$\hat{y}_1$	min ratio
x_6	5/2	1	0	−5/8	0	17/8	20/17 →
x_4	7	0	1	−1/4	0	17/4	28/17
x_2	3/2	0	0	1/8	0	3/8	4
z	$\frac{-5M+9}{8}$	−M	0	$\frac{5M+3}{8}$	1	$\frac{-17M-31}{8}$	

Here the minimum ratio is 20/17 and is associated with the basic variable x_6. So x_6 is replaced by x_1 in the next iteration. Using elementary row operations $\hat{y}_1$ is converted to $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and the same operations are performed in

$\hat{B}^{-1}$. This gives the new $\hat{B}^{-1}$ as follows

$$\therefore \text{ New } \hat{B}^{-1} = \begin{bmatrix} 8/17 & 0 & -5/17 & 0 \\ -2 & 1 & 1 & 0 \\ -3/17 & 0 & 4/17 & 0 \\ 31/17 & 0 & -13/17 & 1 \end{bmatrix}$$

The next BFS is given by

$$\hat{x}_B = \begin{bmatrix} x_1 \\ x_4 \\ x_2 \\ z \end{bmatrix} = \hat{B}^{-1}b = \begin{bmatrix} 8/17 & 0 & -5/17 & 0 \\ -2 & 1 & 1 & 0 \\ -3/17 & 0 & 4/17 & 0 \\ 31/17 & 0 & -13/17 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 12 \\ 0 \end{bmatrix} = \begin{bmatrix} 20/17 \\ 2 \\ 18/17 \\ 154/17 \end{bmatrix}$$

$$\therefore x_B = \begin{bmatrix} x_1 \\ x_4 \\ x_2 \end{bmatrix} = \begin{bmatrix} 20/17 \\ 2 \\ 18/17 \end{bmatrix} \text{ and } z = 154/17.$$

The net evaluations are given by

$$\begin{bmatrix} c_B B^{-1} & 1 \end{bmatrix} \begin{bmatrix} A \\ -c \end{bmatrix} = \begin{bmatrix} 31/17 & 0 & -13/17 & 1 \end{bmatrix} \begin{bmatrix} 4 & 5 & -1 & 0 & 0 & 1 \\ 5 & 2 & 0 & 1 & 0 & 0 \\ 3 & 8 & 0 & 0 & 1 & 0 \\ -5 & -3 & 0 & 0 & 0 & M \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -31/17 & 0 & -13/17 & \frac{31}{17} + M \end{bmatrix}$$

Since there are negative net evaluations, the BFS obtained is not optimal. The most negative $z_j - c_j$ is $z_3 - c_3 = -31/17$. So x_3 is the next incoming basic variable.

Now we compute

$$\hat{y}_3 = \hat{B}^{-1}\hat{a}_3 = \begin{bmatrix} 8/17 & 0 & -5/17 & 0 \\ -2 & 1 & 1 & 0 \\ -3/17 & 0 & 4/17 & 0 \\ 31/17 & 0 & -13/17 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -8/17 \\ 2 \\ 3/17 \\ -31/17 \end{bmatrix}$$

These results are shown in the following revised simplex table.

Basic variables	values	$\hat{B}^{-1}$				$\hat{y}_3$	min ratio
x_1	20/17	8/17	0	-5/17	0	-8/17	...
x_4	2	-2	1	1	0	[2]	1 →
x_2	18/17	-3/17	0	4/17	0	3/17	6
z	154/17	31/17	0	-13/17	1	-31/17	

Obviously x_4 will be replaced by x_3.

Using elementary row operations $\hat{y}_3$ is converted to $\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$ and the same operations are used on $\hat{B}^{-1}$. This gives new $\hat{B}^{-1}$ as follows.

$$\text{New } \hat{B}^{-1} = \begin{bmatrix} 0 & 4/17 & -1/17 & 0 \\ -1 & 1/2 & 1/2 & 0 \\ 0 & -3/34 & 5/34 & 0 \\ 0 & 31/34 & 5/34 & 1 \end{bmatrix}$$

The next BFS is given by

$$\hat{x}_B = \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ z \end{bmatrix} = \hat{B}^{-1}\hat{b} = \begin{bmatrix} 0 & 4/17 & -1/17 & 0 \\ -1 & 1/2 & 1/2 & 0 \\ 0 & -3/34 & 5/34 & 0 \\ 0 & 31/34 & 5/34 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 12 \\ 0 \end{bmatrix} = \begin{bmatrix} 28/17 \\ 1 \\ 15/17 \\ 185/17 \end{bmatrix}$$

$$\therefore \quad x_B = \begin{bmatrix} x_1 \\ x_3 \\ x_2 \end{bmatrix} = \begin{bmatrix} 28/17 \\ 1 \\ 15/17 \end{bmatrix} \text{ and } z = 185/17.$$

The net evaluations are given by

$$\left[c_B B^{-1} \; . \; 1\right]\begin{bmatrix} A \\ -c \end{bmatrix} = \begin{bmatrix} 0 & 31/34 & 5/34 & 1 \end{bmatrix} \begin{bmatrix} 4 & 5 & -1 & 0 & 0 & 1 \\ 5 & 2 & 0 & 1 & 0 & 0 \\ 3 & 8 & 0 & 0 & 1 & 0 \\ -5 & -3 & 0 & 0 & 0 & M \end{bmatrix}$$

$= [0 \quad 0 \quad 0 \quad 31/34 \quad 5/34 \quad M]$

Here all net evaluations are found to be non-negative. Hence we have obtained the optimal solution. The optimal solution is given by

$x_1 = 28/17$, $x_2 = 15/17$ and $z_{max} = 185/17$

101.9 SUMMARY

Revised simplex method is an efficient method and is very useful for large problem. Only necessary part of the simplex table is calculated to pass from one table to the next table. Standard form of the revised simplex method is derived and computational procedure of revised simplex method is discussed. Next utility of revised simplex method is noted and is compared with simplex method. Finally, the method is used to solve some examples.

101.10. SELF ASSESMENT QUESTIONS

Use revised simplex method to solve the following LPP

1. Maximize $z = 5x_1 + 3x_2$

   subject to $3x_1 + 5x_2 \leq 15$

   $5x_1 + 2x_2 \leq 10$

   $x_1, x_2 \geq 0$

   [Ans : $x_1 = \frac{20}{19}$, $x_2 = \frac{45}{19}$, $z_{max} = \frac{235}{19}$ **]**

2. Maximize $z = x_1 + x_2 + 3x_3$

   subject to $3x_1 + 2x_2 + x_3 \leq 1$

   $2x_1 + x_2 + 2x_3 \leq 2$

   $x_1, x_2, x_3 \geq 0$

   [Ans : $x_1 = 0, x_2 = 0, x_3 = 1, z_{max} = 3$**]**

3. Maximize $z = 3x_1 + 5x_2$

   subject to $x_1 \leq 4$

   $x_2 \leq 6$

   $3x_1 + 2x_2 \leq 18$

   $x_1, x_2 \geq 0$

   [Ans : $x_1 = 2, x_2 = 6, z_{max} = 36$**]**

4. Maximize $z = 6x_1 - 2x_2 + 3x_3$

subject to $2x_1 - x_2 + 2x_3 \leq 2$

$x_1 + 4x_3 \leq 4$

$x_1, x_2, x_3 \geq 0$

[Ans : $x_1 = 4, x_2 = 6, x_3 = 0, z_{max} = 12$]

5. Maximize $z = 3x_1 + 2x_2 + 5x_3$

subject to $x_1 + 2x_2 + x_3 \leq 430$

$3x_1 + 2x_3 \leq 460$

$x_1 + 4x_2 \leq 420$

$x_1, x_2, x_3 \geq 0$

[Ans : $x_1 = 0, x_2 = 100, x_3 = 230, z_{max} = 1350$]

6. Minimize $z = x_1 + x_2$

subject to $x_1 + 2x_2 \geq 7$

$4x_1 + x_2 \geq 6$

$x_1, x_2 \geq 0$

[Ans : $x_1 = 5/7, x_2 = 22/7, z_{max} = 27/7$]

7. Minimize $z = 2x_1 + x_2$

subject to $3x_1 + x_2 \leq 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 3$

$x_1, x_2 \geq 0$

[Ans : $x_1 = 3/5, x_2 = 6/5, z_{max} = 12/5$]

8. Maximize $z = 10x_1 + 9x_2$

subject to $8x_1 + 15x_2 \geq 10$

$10x_1 + 6x_2 \leq 10$

$6x_1 + 24x_2 \leq 12$

$x_1, x_2 \geq 0$

[Ans : $x_1 = 14/17, x_2 = 5/17, z_{max} = 185/17$]

9. Maximize $z = 4x_1 + 3x_2$

subject to $3x_1 + 4x_2 \leq 12$

$3x_1 + 3x_2 \leq 10$

$2x_1 + x_2 \leq 4$

$x_1, x_2 \geq 0$

[Ans : $x_1 = 4/5, x_2 = 12/5, z_{max} = 52/5$]

10. Minimize $z = 2x_3 - 3x_4 - x_5$

subject to $x_1 - 2x_3 + x_4 + x_5 = 4$

$x_2 + 3x_3 - x_4 + 2x_5 = 2$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

[Ans : $x_1 = x_2 = 0, x_3 = 6, x_4 = 16, x_5 = 0, z_{min} = -36$]

11. Minimize $z = 2x_1 + 4x_2 + x_3$

subject to $x_1 + 2x_2 - x_3 \leq 5$

$2x_1 - x_2 + 2x_3 = 2$

$-x_1 + 2x_2 + 2x_3 \geq 1$

$x_1, x_2, x_3 \geq 0$

[Ans : $x_1 = 1/3, x_2 = 0, x_3 = 2/3, z_{min} = 4/3$]

12. Maximize $z = x_1 + 2x_2 + 3x_3 - x_4$

subject to $x_1 + 2x_2 + 3x_3 = 15$

$2x_1 + x_2 + 5x_3 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$

$x_1, x_2, x_3, x_4 \geq 0$

[Ans : $x_1 = 5/2, x_2 = 5/2, x_3 = 5/2, x_4 = 0, z_{max} = 15$]

13. Maximize $z = -x_1 - 2x_2 - 4x_3 - 3x_4$

subject to $2x_1 + x_2 + 3x_3 + 4x_4 \geq 10$

$x_1 + 3x_2 + 2x_3 + 5x_4 \geq 20$

$3x_1 + 2x_2 + x_3 + x_4 \geq 15$

$x_1, x_2, x_3, x_4 \geq 0$

[Ans : $x_1 = 5/4, x_2 = 75/14, x_3 = 0, x_4 = 15/28, z_{max} = -96/7$]

101.11. Suggested Further Readings

1. Operations Research : Kanti Swarup, P. K. Gupta, Man Mohan Sultan Chand & Sons.
2. Linear Programming and Game Theory : Chakravorty & Ghosh, Moulik Library
3. Operations Research : J. K. Sharma, Macmillan India Limited
4. Linear Programming and Theory of Games : S. D. Sharma and Hemlata Sharma; Keder Nath Ram Nath & Co.
5. Operations Research : R. K. Gupta, Krishna Prakashan Mandir
6. Linear Programming and Theory of Games : P. K. Gpupa, Khanna Publishers.
7. Operations Research : Hamdy A. Taha, Prentice Hall of India Private Limited.
8. Operations Research : C. K. Mustafi, New Age International Ltd.

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

Part - II

Paper - IX (Special paper)

OPERATIONS RESEARCH

Module No. 102

(Dual Simplex Method Bounded Variables LPP, and Decomposition Principle)

STRUCTURE :

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102.2 Objective
102.3 Comparison between simplex and Dual Simplex method
102.4 Applications of the dual simplex method
102.5 Criteria for incoming and outgoing basic variable
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DUAL SIMPLEX METHOD

102.1 INTRODUCTION

The dual simplex method was first discovered by CE Lemke, a student of charnes. The dual simplex method is used to solve problems which is optimal but infeasible. In simplex method we start with a basic feasible but non-optimal solution and works fowards optimality. In dual simplex method we start with a basic infeasible solution which is optimal and works towards feasibility. The two methods are quite similar, the only difference lies in the criterion used for selecting a variable to enter the basis and to leave the basis. Further, in the dual simplex

method we first determine the variable to leave the basis and then the variable to enter the basis, while reverse is done in the case of simplex method.

The dual simplex method gives an algorithm in which we start with a basic optimal solution of the primal in which all $z_j - c_j \geq 0$ but not feasible i.e. some basic solution are negative. At each iteration the number of negative basic variables are decreased while maintaining the optimality. An optimal solution is reached in a finite number of steps. The benifit of this procedure lies in the fact that we need not take the help of any artificial variable and hence it reduces a lot of labour.

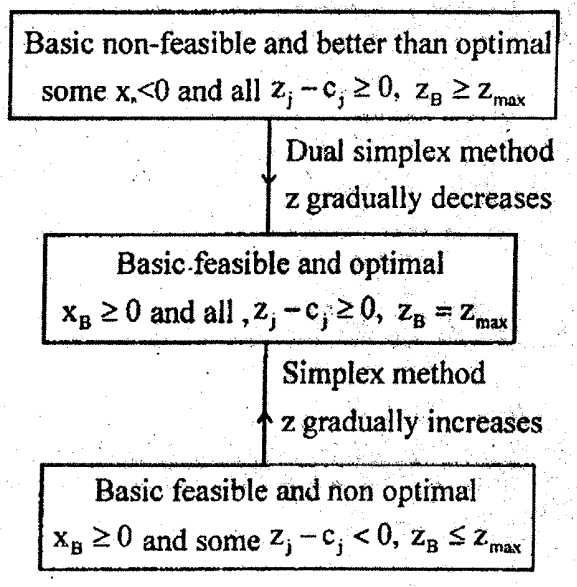
102.2 OBJECTIVE

A large number of LPP can be solved by an alternative method called dual simplex method. The method has a wide application in different useful areas like sensitivity analysis, parametric programming, integer programming and some non linear programming problems. As no artificial variables are needed, the size of the tables remains small in this method. To broaden the application area this method is also modified. Another class of LPP where variables are bounded are handled by a method known as Method of Bounded Variables. These useful methods are also illustraled by examples.

102.3 Comparison between Simplex method and Dual Simplex method

In simplex method the initial solution is basic feasible and non optimal. In subsequent tables the value of the objective function gradually increases and finally reaches to its optimal value. In each table the solution is basic feasible and non-optimal.

In dual simplex method the initial solution is basic non-feasible and optimal. In subsequent tables the value of the objective function gradually decreases and finally reaches to its optimal value. In each table the solution is basic non-feasible and optimal (or better than optimal).



102.4 Applications of the dual simplex method

If the given LPP is optimal and infeasible then only dual simplex method is applicable. For many practical problem the initial table does not satisfy these conditions and as a consequence dual simplex method can not be applied. Simplex method has no such restriction and is applicable to any LPP. Hence as a rule the regular simplex method is preferred over the dual simplex method for solving the general LPP. However, there are instances when the dual simplex method has a distinct advantage over the regular simplex method. There are problems in which a dual feasible table is readily available to start the dual simplex method and for such problems the optimal BFS is obtained easily in comparison to simplex method. Some of the applications of dual simplex method are

i) Sensitivity analysis when the right hand side vector be changed or when new constraints are added
ii) Parametric programming
iii) Integer programming problem
iv) Some non-linear programming problem

Theorem 102.1: If for a basic solution (not necessarily feasible) all $z_j - c_j \geq 0$ then the value of the objective function corresponding to this basic solution is optimal or better than optimal

Proof: Let the LPP be

Maximize $z = cx$

subject to $Ax = b,\ x \geq 0$

Where $c,\ x^T \in R^n,\ b^T \in R^m$ and A is a real m×n matrix. Let $x_B = B^{-1}b$ be a basic solution of this LPP and all $z_j - c_j \geq 0$. The dual of this LPP is given by

Minimize $z^* = b^T\omega$

subject to $A^T\omega \geq c^T,$

w is unrestricted

The value of z corresponding to the basic solution x_B is given by

$$z_B = c_B x_B = \left(c_B x_B\right)^T = x_B^T c_B^T = \left(B^{-1}b\right)^T c_B^T$$

$$= b^T\left(B^{-1}\right)^T c_B^T = b^T\left[\left(B^{-1}\right)^T c_B^T\right] = b^T\left(c_B B^{-1}\right)^T = b^T\omega_1$$

where $\omega_1 = \left(c_B B^{-1}\right)^T$

We note that $\omega_1^T \in R^m$.

Since all $z_j - c_j \geq 0$ we have

$$\left[c_B B^{-1} \quad I\right]\begin{bmatrix} A \\ -c \end{bmatrix} \geq 0$$

or, $(c_B B^{-1})A - c \geq 0$

or, $(c_B B^{-1})A \geq c$

or, $(c_B B^{-1} A)^T \geq c^T$

or, $A^T(c_B B^{-1})^T \geq c^T$

or, $A^T \omega_1 \geq c^T$

$\therefore$ ω_1 is a BFS of the dual LPP.

Now $b^T\omega_1$ is the value of the dual ofjective function for the solution ω_1

$\therefore$ $z^*_{min} \leq b^T\omega_1$

But we have $b^T\omega_1 = z_B$.

$\therefore$ $z^*_{min} \leq z_B$

From the theorem of duality we have $z^*_{min} = z_{max}$

Hence $z_{max} \leq z_B$ or, $z_B \geq z_{max}$

i.e. the value of the objective function corresponding to the basic solution $x_B = B^{-1}b$ is equal to or more than the optimal value of the objective function.

Theorem 102.2 : Let x_{B_r} be a negative basic variable in a dual simplex table and all net evaluations $z_j - c_j$ be non-negative and the primal LPP is of maximization. If $y_{rj} \geq 0$ for all non basic variables x_j then there does not exist any feasible solution to the primal LPP.

Let the primal LPP be

Maximize $z = cx$

subject to $Ax = b,\ x \geq 0$

where $c,\ x^T \in R^n,\ b^T \in R^m$ and A is a real m×n matrix.

The dual of this LPP is

Minimize $z' = b^T\omega$

subject to $A^T\omega \geq c^T$

ω is unrestricted and $\omega^T \in R^m$

Since all $z_j - c_j \geq 0$ we have

$$\left[c_B B^{-1} \quad 1\right]\begin{bmatrix} A \\ -c \end{bmatrix} \geq 0$$

or, $A^T\omega_1 \geq c^T$ where $\omega_1 = \left(c_B B^{-1}\right)^T$ ..... (1)

$\therefore$ ω_1 is a feasible solution of the dual LPP.

Let θ be any positive real number and $\omega_0 = \omega_1 + \theta\beta_r$ where β_r is the rth column of $(B^{-1})^T$.

Now $A^T\omega_0 = A^T(\omega_1 + \theta\beta_r) = A^T\omega_1 + \theta A^T\beta_r$ ..... (2)

we have $y_j = B^{-1}a_j$ for all j.

$\therefore$ $Y = B^{-1}A$ or, $Y^T = \left(B^{-1}A\right)^T = A^T\left(B^{-1}\right)^T$

or, $Y^T = A^T[\beta_1\ \beta_2\ \ldots.\ \beta_m]$

$= \left[A^T\beta_1\ A^T\beta_2\ \ldots..\ A^T\beta_m\right]$

or, $\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{bmatrix}^T = \left[A^T\beta_1\ A^T\beta_2\ \ldots..\ A^T\beta_m\right]$

or, $\begin{bmatrix} y_{11} & y_{21} & \cdots & y_{m1} \\ y_{12} & y_{22} & \cdots & y_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ y_{1n} & y_{2n} & \cdots & y_{mn} \end{bmatrix} = \left[A^T\beta_1\ A^T\beta_2\ \cdots\ A^T\beta_m\right]$

$\therefore$ $\begin{bmatrix} y_{r1} \\ y_{r2} \\ \vdots \\ y_m \end{bmatrix} = A^T\beta_r$ for r = 1, 2, ..., m.

Since all $y_{rj} \geq 0$, we have $A^T\beta_r \geq 0$

$\therefore$ As $\theta \geq 0$ we have $\theta A^T\beta_r \geq 0$.

Thus from (2) we have

$$A^T\omega_0 \geq A^T\omega_1$$

Using (1) this gives $A^T\omega_0 \geq c^T$.

Hence ω_0 is a feasible solution to the dual LPP for all θ.

The value of the dual objective function z^1 for the feasible solution ω_0 is given by

$$z^1_{\omega_0} = b^T\omega_0 = b^T(\omega_1 + \theta\beta_r) = b^T\omega_1 + \theta b^T\beta_r \quad \ldots\ldots (3)$$

From $x_B = B^{-1}b$ we have

$$x_B^T = (B^{-1}b)^T \text{ or, } x_B^T = b^T(B^{-1})^T$$

$$\text{or, } \left[x_{B_1}\ x_{B_2} \ldots x_{B_r} \ldots x_{B_m}\right] = b^T[\beta_1\ \beta_1 \ldots \beta_r \ldots \beta_m]$$

$\therefore x_{B_r} = b^T\beta_r$ for all r.

Thus from (3) we have

$$z^1_{\omega_0} = b^T\omega_1 + \theta x_{B_r}.$$

Letting $\theta \to \infty$ we get $z^1_{\omega_0} \to -\infty$ $\left[\because x_{B_r} < 0 \text{ and } \theta > 0\right]$.

Thus $z^1_{\omega_0}$ decreases without bound when $\theta \to \alpha$. i.e. there exists an unbounded solution to the dual LPP. This shows that the primal LPP has no feasible solution.

102.5 Criteria for incoming and outgoing basic variable in dual simplex method.

In the transmation formula of simplex method if the basic variable x_{B_r} is replaced by the non-basic variable x_k then we have

$$\hat{z} = z - \frac{x_{Br}}{y_{rk}}(z_k - c_k) \quad \ldots\ldots (1)$$

$$\text{and } \left(\hat{z}_j - \hat{c}_j\right) = \left(z_j - c_j\right) - \frac{y_{rj}}{y_{rk}}\left(z_k - c_k\right) \quad \ldots\ldots (2)$$

As we want to remove negative basic variables, we choose the most negative basic variable x_{B_r} (say) as

outgoing basic variable from the list of all basic variables.

We know that if for some basic solution (not necessarily feasible) all components of net evaluations are non-negative then the value of the objective function to this basic solution is optimal or better than optimal. So we intend to lower down the value of the objective function to get z_{max}. For this from (1) we should have $y_{rk} < 0$ as $x_{B_r} < 0$ and $z_k - c_k \geq 0$. This should be one criterion for incoming basic variable.

In the next table we want the solution to be optimal or better than optimal. So we should have $\hat{z}_j - \hat{c}_j \geq 0$ for all j.

$\therefore$ From (2) we have

$$z_j - c_j - \frac{y_{rj}}{y_{rk}}(z_k - c_k) \geq 0 \quad \text{for all } j$$

$$\text{or, } z_j - c_j \geq \frac{y_{rj}}{y_{rk}}(z_k - c_k) \quad \text{for all } j \qquad \ldots\ldots\ldots\ldots (3)$$

When $y_{rj} \geq 0$ then (3) is satisfied as $y_{rk} < 0$ and all $z_j - c_j \geq 0$.

When $y_{rj} < 0$ then (3) is satisfied

$$\text{if } \frac{z_j - c_j}{y_{rj}} \leq \frac{z_k - c_k}{y_{rk}} \quad \text{for all } j$$

Hence we are to choose k such that

$$\max_{y_{rj} < 0}\left\{\frac{z_j - c_j}{y_{rj}}\right\} = \frac{z_k - c_k}{y_{rk}}$$

102.6 Dual simplex Algorithm

The iterative procedure for dual simplex algorithm are as follows :

Step 1. Convert the minimization LPP into that of maximization if it is in the minimization form.

Step 2. Convert the $\geq$ type inequations, representing the constraints of the given LPP, if any into those of $\leq$ type by multiplying the corresponding constraints by - 1.

Step 3. Introduce slack variables in the constraints of the given LPP and obtain an initial basic solution. Put this solution in the starting dual simplex table.

Step 4. Test the nature of the net evaluations $z_j - c_j$ in the starting simplex table.

i) If all $z_j - c_j$ and x_{B_i} are non-negative for all i and j, then an optimum basic feasible solution has been obtained.

ii) If all $z_j - c_j$ are non-negative and at least one basic variable, say x_{B_r}, is negative then go to step 5.

iii) If at least one $z_j - c_j$ is negative then dual simplex method is not applicable. In this case we are to apply artificial constraint method.

Step 5. Select the most negative basic variable, say x_{B_k}, as outgoing basic variable.

Step 6. Test the nature of all $y_{kj}, j = 1, 2, ..., n$.

i) If all y_{kj} are non-negative, there does not exist any feasible solution to the given LPP.

ii) If at least one y_{kj} is negative, then compute the replacement ratios

$$\left\{\frac{z_j - c_j}{y_{kj}} : y_{kj} < 0\right\}, j = 1, 2, ..., n$$

and choose the maximum of these. If the maximum of these be $\frac{z_r - c_r}{y_{kr}}$ then x_r is the incoming basic variable. i.e. x_{B_k} is replaced by x_r.

Step 7. With y_{kr} as the key element form the next table. Using elementary row operation convert the key element to unity and all other elements of the key column to zero to get the improved solution.

Step 8. Repeat the steps 4 to 7 until either an optimum basic feasible solution is obtained or there is an indication of no feasible solution.

102.7 Illustrative Examples

Example 102.7.1 : Solve the following LPP by dual simplex method

Minimize $z = 2x_1 + x_2$

subject to $3x_1 + x_2 \geq 3$

$4x_1 + x_2 \geq 6$

$x_1 + 2x_2 \geq 3$

$x_1, x_2 \geq 0$

Solution : Converting the given LPP into maximization and changing all $\geq$ type inequations to $\leq$ type and finally adding slack variables $x_3 \geq 0$, $x_4 \geq 0$, $x_5 \geq 0$, the reformulated LPP in its standard form becomes

Maximize $z^1 = -2x_1 - x_2 + 0x_3 + 0x_4 + 0x_5$

subject to

$$\begin{aligned} -3x_1 - x_2 + x_3 &= 3 \\ -4x_1 - x_2 + x_4 &= 6 \\ -x_1 - 2x_2 + x_5 &= 3 \end{aligned}$$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

The solution of this LPP by dual simplex method is shown in the following tables.

	c_j	−2	−1	0	0	0
c_B y_B	x_B	y_1	y_2	y_3	y_4	y_5
0 y_3	−3	−3	−1	1	0	0
0 y_4	−6	[−4]	−1	0	1	0 →
0 y_5	−3	−1	−2	0	0	1
$z' = 0$	$z_j - c_j$	2	1	0	0	0
$\frac{z_j - c_j}{y_{2j}} : y_{2j} < 0$		$\frac{2}{-4} = -1/2$	$\frac{1}{-1} = -1$			
0 y_3	3/2	0	−1/4	1	−3/4	0
−2 y_1	3/2	1	1/4	0	−1/4	0
0 y_5	−3/2	0	[−7/4]	0	−1/4	1 →
$z' = -3$	$z_j - c_j$	0	1/2	0	1/2	0
$\frac{z_j - c_j}{y_{3j}} : y_{3j} < 0$			$(1/2)/(-7/4) = -2/7$			
0 y_3	12/7	0	0	1	−5/7	−1/7
−2 y_1	9/7	1	0	0	−2/7	1/7
−1 y_2	6/7	0	1	0	1/7	−4/7
$z' = -24/7$	$z_j - c_j$	0	0	0	3/7	2/7

In the first table $\max\left\{-\frac{1}{2}, -1\right\} = -\frac{1}{2}$ and is associated with y_1. $\therefore$ y_4 is replaced by y_1 for the second table.

In the second table there is only one ratio $-2/7$ and is associated with y_2. $\therefore$ y_5 is replaced by y_2 for the third table.

In the third table all x_{B_i} are non negative. So this is optimal table. The optimal solution is $x_1 = 9/7$, $x_2 = 6/7$ and $z'_{max} = -24/7$ $\therefore$ $z_{min} = -z'_{max} = 24/7$.

Example 102.7.2 : Solve the following LPP by dual simplex method

Maximize $z = -2x_1 - 2x_2 - 4x_3$

subject to $2x_1 + 3x_2 + 5x_3 \le 2$

$3x_1 + x_2 + 2x_3 \ge 3$

$$x_1 + 4x_2 + 6x_3 \geq 5$$
$$x_1, x_2, x_3 \geq 0$$

Solution: Converting the $\geq$ type inequations into $\leq$ type and introducing the slack variables $x_4 \geq 0$, $x_5 \geq 0$, $x_6 \geq 0$, the given LPP can be written in the standard form as

Maximize $z = -2x_1 - 2x_2 - 4x_3 + 0x_4 + 0x_5 + 0x_6$

subject to
$$2x_1 + 3x_2 + 5x_3 + x_4 = 2$$
$$-3x_1 - x_2 - 2x_3 + x_5 = -3$$
$$-x_1 - 4x_2 - 6x_3 + x_6 = -5$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

The following tables are obtained by using dual simplex method to this LPP.

	c_j		–2	–2	–4	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_4	2	2	3	5	1	0	0
0	y_5	–3	–3	–1	–2	0	1	0
0	y_6	–5	–1	[–4]	–6	0	0	1
$z = 0$		$z_j - c_j$	2	2	4	0	0	0
$\frac{z_j - c_j}{y_{3j}} : y_{3j} < 0$			2/(–1) = –2	2/(–4) = –1/2	4/(–6) = –2/3			
0	y_4	–7/4	5/4	0	1/2	1	0	3/4
0	y_5	–7/4	[–11/4]	0	–1/2	0	1	–1/4
–2	y_2	5/4	1/4	1	3/2	0	0	–1/4
$z = -5/2$		$z_j - c_j$	3/2	0	1	0	0	1/2
$\frac{z_j - c_j}{y_{2j}} : y_{2j} < 0$			(3/2)/(–11/4) = –6/11		1/(–1/2) = –2			(1/2)/(–1/4) = –2
0	y_4	–28/11	0	0	3/11	1	5/11	7/11
–2	y_1	7/11	1	0	2/11	0	–4/11	1/11
–2	y_2	12/11	0	1	16/11	0	1/11	–3/11
$z = -38/11$		$z_j - c_j$	0	0	8/11	0	6/11	4/11

In the first table $x_6 = -5$ is the most negative basic variable and $\max\left\{-2, -\frac{1}{2}, -\frac{2}{3}\right\} = -\frac{1}{2}$ which is associated with the non basic variable x_2. So x_6 is replaced by x_2. In the second table $x_4 = -7/4$, $x_5 = -7/4$ are the most negative basic variables. We choose x_5 arbitrarily. Here $\max\{-6/11, -2, -2\} = -6/11$ which is associated with the non basic variable x_1. So x_5 is replaced by x_1. In the third table $x_{B_1} = x_4 < 0$ and all $y_{1j} \geq 0$.

$\therefore$ The given LPP has no feasible solution.

102.8. Modification of dual simplex method

If the initial table of the dual simplex method contains some negative basic variables and some of the net evaluations are negative then the dual simplex method is not applicable. In such situation dual simplex method is to be modified to form an equivalent LPP in which some basic variables are negative but all net evaluations are non-negative. Hence standard dual simplex method can be applied to that equivalent LPP.

The artificial constraint is one such method. In this method we consider the variables corresponding to which the net evaluations are negative and the variable corresponding to the most negative component of net evaluations is noted. Let $z_p - c_p$ be the most negative net evaluation. So we consider the corresponding variable x_p. In this method we have to consider the artificial constraint $\Sigma x_j \leq M$ where Σ is extended over all j's for which $z_j - c_j < 0$ and M is a sufficiently large positive number. Adding slack variable x_M to this constraint we get

$$\Sigma x_j + x_M = M$$

From this we find x_p as

$$x_p = M - \left(x_M + \sum_{j \neq p} x_j \right)$$

This x_p is then substituted in the original objective function and in the set of all constraints. This new problem together with the new added artificial constraint is equivalent to the given problem. This equivalent LPP will have all $z_j - c_j \geq 0$. Thus dual simplex method can be applied.

102.9. ILLUSTRATIVE EXAMPLES

Example 102.9.1 Use the artificial constraint method to find the initial basic solution of the following problem and then apply the dual simplex algorithm to solve it

$$\begin{aligned} \text{Minimize} \quad & z = -2x_1 - x_2 - x_3 \\ \text{subject to} \quad & 4x_1 + 6x_2 + 3x_3 \leq 8 \\ & -x_1 + 9x_2 - x_3 \geq 3 \\ & 2x_1 + 3x_2 - 5x_3 \geq 4 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution : We first convert the minimization problem to maximization and then change the inequation of $\geq$ type

into $\leq$ type. Finally adding slack variables $x_4 \geq 0$, $x_5 \geq 0$, $x_6 \geq 0$, we get the standard form of LPP in dual simplex method as

$$\begin{aligned}
\text{Maximize} \quad & z^1 = 2x_1 + x_2 + x_3 + 0x_4 + 0x_5 + 0x_6 \\
\text{subject to} \quad & 4x_1 + 6x_2 + 3x_3 + x_4 = 8 \\
& x_1 - 9x_2 + x_3 + x_5 = -3 \\
& -2x_1 - 3x_2 + 5x_3 + x_6 = -4 \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.
\end{aligned}$$

The initial dual simplex table is

		c_j	2	1	1	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_4	8	4	6	3	1	0	0
0	y_5	−3	1	−9	1	0	1	0
0	y_6	−4	−2	−3	5	0	0	1
$z' = 0$		$z_j - c_j$	−2	−1	−1	0	0	0

Here there are negative net evaluations, so standard dual simplex method is not applicable.

The negative net evaluations are $z_1 - c_1, z_2 - c_2, z_3 - c_3$, & most negative net evaluation is $z_1 - c_1 = -2$.

$\therefore$ The artificial constraint is

$x_1 + x_2 + x_3 \leq M$ where M is a very large positive number. Adding slack variable x_M we have

$x_1 + x_2 + x_3 + x_M = M$

From this we have $x_1 = M - x_2 - x_3 - x_M$ using this in the LPP and adding the artificial constraint. We have

$$\begin{aligned}
\text{Maximize} \quad & z^1 = 2(M - x_2 - x_3 - x_M) + x_2 + x_3 + 0x_4 + 0x_5 + 0x_6 \\
\text{subject to} \quad & 4(M - x_2 - x_3 - x_M) + 6x_2 + 3x_3 + x_4 = 8 \\
& (M - x_2 - x_3 - x_M) - 9x_2 + x_3 + x_5 = -3 \\
& -2(M - x_2 - x_3 - x_M) - 3x_2 + 5x_3 + x_6 = -4 \\
& x_1 + x_2 + x_2 + x_M = M \\
& x_1, x_2, x_3, x_4, x_5, x_6, x_M \geq 0
\end{aligned}$$

or,

$$\begin{aligned}
\text{Maximize} \quad & z^1 = -2x_M - x_2 - x_3 + 0x_4 + 0x_5 + 0x_6 + 2M \\
\text{subject to} \quad & -4x_M + 2x_2 - x_3 + x_4 = 8 - 4M \\
& -x_M - 10x_2 + x_5 = -3 - M \\
& 2x_M - x_2 + 7x_3 + x_6 = -4 + 2M \\
& x_M + x_1 + x_2 + x_3 = M \\
& x_1, x_2, x_3, x_4, x_5, x_6, x_M \geq 0
\end{aligned}$$

The following tables are obtained using simplex method.

		c_j	−2	0	−1	−1	0	0	0
c_B	y_B	x_B	y_M	y_1	y_2	y_3	y_4	y_5	y_6
0	y_4	$8-4M$	$\boxed{-4}$	0	2	−1	1	0	0
0	y_5	$-3-M$	−1	0	−10	0	0	1	0
0	y_6	$-4+2M$	2	0	−1	7	0	0	1
0	y_1	M	1	1	1	1	0	0	0
$z'=0$		z_j-c_j	2	1	1	1	0	0	0
$\frac{z_j-c_j}{y_{1j}}:y_{1j}<0$			−1/2			−1			
−2	y_M	$-2+M$	1	0	−1/2	1/4	−1/4	0	0
0	y_5	−5	0	0	$\boxed{-21/2}$	1/4	−1/4	1	0
0	y_6	0	0	0	0	13/2	1/2	0	1
0	y_1	2	0	1	3/2	3/4	1/4	0	0
$z'=4-2M$		z_j-c_j	0	0	2	1/2	1/2	0	0
$\frac{z_j-c_j}{y_{2j}}:y_{2j}<0$					−4/21		−2		
−2	y_M	$M-\frac{37}{21}$	1	0	0	5/21	−5/21	−1/21	0
−1	y_2	10/21	0	0	1	−1/42	1/42	−2/21	0
0	y_6	0	0	0	0	13/2	1/2	0	1
0	y_1	9/7	0	1	0	11/14	3/14	1/7	0
$z'=-2M+\frac{64}{21}$		z_j-c_j	0	0	0	13/42	19/42	4/21	0

Here all basic variables are non-negative. So this is the optimal table. The optimal solution is $x_1=9/7$, $x_2=10/21$, $x_3=0$ and $z^1_{max}=(-2M+64/21)+2M=\frac{64}{21}$. Therefore $z_{min}=-z^1_{max}=-64/21$.

Example 102.9.2 Use the artificial constraint method to find the initial basic solution of the following problem and then apply the dual simplex algorithm to solve it :

Maximize $z=2x_1-3x_2-2x_3$

subject to $x_1 - 2x_2 - 3x_3 = 8$

$2x_2 + x_3 \le 10$

$x_2 - 2x_3 \ge 4$

$x_1, x_2, x_3 \ge 0$

Solution : We first change the inequation of $\ge$ type into $\le$ type. Adding slack variable $x_4 \ge 0$, $x_5 \ge 0$ we get the standard form of the LPP in dual simplex method as

Maximize $z = 2x_1 - 3x_2 - 2x_3$

subject to $x_1 - 2x_2 - 3x_3 = 8$

$2x_2 + x_3 + x_4 = 10$

$-x_2 + 2x_3 + x_5 = -4$

$x_1, x_2, x_3, x_4, x_5 \ge 0$

The initial dual simplex table is

		c_j	2	–3	–2	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
2	y_1	8	1	–2	–3	0	0
0	y_4	10	0	2	1	1	0
0	y_5	–4	0	–1	2	0	1
$z = 16$		$z_j - c_j$	0	–1	–4	0	0

Since these are negative net evaluations, standard dual simplex method is not applicable. The negative net evaluations are $z_2 - c_2$ and $z_3 - c_3$ and most negative net evaluation is $z_3 - c_3 = -4$.

$\therefore$ The artificial constraint is $x_2 + x_3 \le M$ where M is a very large positive number. Adding slack variable x_M we have

$x_2 + x_3 + x_M = M$

From this we have $x_3 = M - x_2 - x_M$.

Using this in the LPP and adding the artificial constraint we have the equivalent LPP as

Maximize $z = 2x_M + 2x_1 - x_2 - 2M$

subject to $3x_M + x_1 + x_2 = 3M + 8$

$-x_M + x_2 + x_4 = -M + 10$

$-2x_M - 3x_2 + x_5 = -2M - 4$

$x_M + x_2 + x_3 = M$

$x_M, x_1, x_2, x_3, x_4, x_5 \ge 0$

The dual simplex tables are as follows.

		c_j	2	2	−1	0	0	0
c_B	y_B	x_B	y_M	y_1	y_2	y_3	y_4	y_5
2	y_1	$3M + 8$	3	1	1	0	0	0
0	y_4	$-M + 10$	−1	0	1	0	1	0
0	y_5	$-2M - 4$	−2	0	**−3**	0	0	1
0	y_3	M	1	0	1	1	0	0
$z = 6M + 16$		$z_j - c_j$	4	0	3	0	0	0
$\frac{z_j - c_j}{y_{3j}} : y_{3j} < 0$			−2		−1			
2	y_1	$\frac{7M + 20}{3}$	7/3	1	0	0	0	1/3
0	y_4	$\frac{-5M + 26}{3}$	**−5/3**	0	0	0	1	1/3
−1	y_2	$\frac{2M + 4}{3}$	2/3	0	1	0	0	−1/3
0	y_3	$\frac{M - 4}{3}$	1/3	0	0	1	0	1/3
$z = 4M + 12$		$z_j - c_j$	2	0	0	0	0	1
$\frac{z_j - c_j}{y_{2j}} : y_{2j} < 0$			−6/5					
2	y_1	$\frac{94}{5}$	0	1	0	0	7/5	4/5
2	y_M	$\frac{5M - 26}{5}$	1	0	0	0	−3/5	−1/5
−1	y_2	$\frac{24}{5}$	0	0	1	0	2/5	−1/5
0	y_3	$\frac{2}{5}$	0	0	0	1	1/5	2/5
$z = 10M + 112$		$z_j - c_j$	0	0	0	0	6/5	7/5

In this table all basic variables are non-negative. So this is the optimal table. The optimal solution is

$x_1 = 94/5,\ x_2 = 24/5,\ x_3 = 2/5$ and $z_{max} = \frac{10M+112}{5} - 2M = \frac{112}{5}$.

102.10. BOUNDED VARIABLES LP PROBLEMS

Very often a linear programming problem may have, in addition to the given constraints, some or all variables with lower and upper limits. In such cases the standard form of LPP appears as

Optimize $z = cx$

subject to $Ax = b$

$l \le x \le u$

where $l = [l_1, l_2, ..., l_n]^T$ and $u = [u_1, u_2, ..., u_n]^T$ denote the lower and upper constraints bounds for variable x respectively,

$x = [x_1, x_2, ..., x_n]^T$, $b = [b_1, b_2, ..., b_m]^T$, $c = [c_1, c_2, ..., c_n]$ and

$A = [a_{ij}]$ $i = 1, 2, ..., m;$ $j = 1, 2, ..., n.$

The inequality constraints $l \le x \le u$ in the LP model can be converted into equality constraints by introducing necessary slack and surplus variables s' and s" as follows

$x \ge l$ becomes $x - s'' = l,\ s'' \ge 0$

$x \le u$ becomes $x + s' = u,\ s' \ge 0.$

Thus for the given LPP, there are now m + 2n constraint equations in 3n variables. However, this problem can be reduced in size by using a special technique which ultimately reduces the constraints to the form Ax = b.

The lower bound constraint can be written as $x = l + s''$ and thus x can be eliminated from all the remaining constraints. The new variables in the problem thus becomes s' and s". As both of these variables are non-negative, x will always be non-negative.

The upper bound constraint can be written as $x = u - s'$. But this does not serve the purpose, since there is no guarantee that x will be non-negative. It is in this context that a special technique known as bounded variable simplex method was developed to overcome this difficulty.

In bounded variable simplex method the optimality condition for a solution is the same as the simplex method. But the inclusion of constraints $x + s' = u$ in the simplex table require modification in the feasibility condition of the simplex method due to the following reasons :

i) A basic variable should become a non-basic variable at its upper bound. We note that in usual simplex method all non-basic variables are at zero level.

ii) When a non-basic variable become basic variable, its value should not exceed its upper bound and also should not disturb the non-negativity and upper bound conditions of all existing basic variables.

102.11 SIMPLEX METHOD FOR BOUNDED VARIABLES

The iterative procedure for determining an optimum basic feasible solution to a LPP with bounded variables are as follows :

Step 1 : i) If the objective function of the given LPP is of minimization, then convert it into that of maximization by using the following relationship

Minimize $z = -$ Maximize z^* where $z^* = -z$.

ii) Check whether all b_i are positive. If any one is negative, then multiply the corresponding constraint by – 1 in order to make it positive.

iii) Express the mathematical model of the given LPP in standard form by adding slack / or surplus variables.

Step 2 : Obtain an initial basic feasible solution. If any of the basic variables is at a positive lower bound, then substitute it out at its lower bound.

Step 3 : Calculate $z_j - c_j$ as usual for all non-basic feasible. Examine values of $z_j - c_j$.

i) If all $z_j - c_j \geq 0$ then the current basic feasible solution is the optimal solution.

ii) If at least one $z_j - c_j < 0$ and this column has at least one entry positive (i.e. $y_{ij} > o$) for some row i, then it indicates that an improvement in the value of objective function z is possible.

Step 4 : If case (ii) of step 3 holds, select a non-basic variable to enter into the new solution according to the following criterion

$$z_k - c_k = \min_i \{z_j - c_j : z_j - c_j < 0\}$$

Step 5 : After identifying the column vector which will enter the basis matrix B, the vector to be removed from B is calculated. For this calculate the quantities

$$\theta_1 = \min_i \left\{ \frac{x_{Bi}}{y_{ir}} : y_{ir} > 0 \right\}$$

$$\theta_2 = \min \left\{ \frac{u_r - x_{B_i}}{-y_{ir}} : y_{ir} < 0 \right\}$$

$$\theta = \min\{\theta_1, \theta_2, u_r\}$$

where u_r is the upper bound for the variable x_r in the current basic feasible solution. Obviously, if all $y_{ir} > 0$ then $\theta_2 = \alpha$.

i) If $\theta = \theta_1$ which corresponds to x_{B_k}, then y_r enters the basis and y_k leaves it, using the regular row operations of the simplex method.

ii) If $\theta = \theta_2$ which corresponds to x_{B_k}, then y_r enters the basis and y_k leaves it, x_{B_k} being not basic at its upper bound must be substituted out by using

$$\left(x_{B_k}\right)_j = \left(x_{B_k}\right)_j - y_{kj}u_j$$

where $0 \le \left(x_{B_k}\right)_j \le u_i$ and $\left(x_{B_k}\right)_j$ denotes that the variable x_j was x_{B_k}.

iii) If $\theta = u_r$, x_r is substituted at its upper bound difference $u_r - x_r$ while remaining non-basic.

Step 6 : Go to step 4 and repeat the procedure until all θ entries in the $z_j - c_j$ row are either negative or zero.

102.12. ILLUSTRATIVE EXAMPLES

Example 102.12.1 : Solve the following LP problem using bounded variable simplex method

$$\text{Maximize} \quad z = 3x_1 + 5x_2 + 2x_3$$

$$\text{subject to} \quad x_1 + 2x_2 + 2x_3 \le 14$$

$$2x_1 + 4x_2 + 3x_3 \le 23$$

$$0 \le x_1 \le 4$$

$$2 \le x_2 \le 5$$

$$0 \le x_3 \le 3$$

Solution : Here the variable x_2 has a positive lower bound. We take $x_2' = x_2 - 2$ or, $x_2 = 2 + x_2'$. Thus the constraint $2 \le x_2 < 5$ becomes $0 \le x_2' \le 3$. Putting $x_2 = 2 + x_2'$ the new LP problem becomes

$$\text{Maximize} \quad z = 3x_1 + 5(x_2' + 2) + 2x_3$$

$$\text{subject to} \quad x_1 + 2(x_2' + 2) + 2x_3 \le 14$$

$$2x_1 + 4(x_2' + 2) + 3x_3 \le 23$$

$$0 \le x_1 \le 4$$

$$0 \le x_2' \le 3$$

$$0 \le x_3 \le 3.$$

Introducing non-negative slack variables s_1 and s_2, the standard form of LP problem becomes

$$\text{Maximize} \quad z = 3x_1 + 5x_2' + 2x_3 + 0s_1 + 0s_2 + 10$$

$$\text{subject to} \quad x_1 + 2x_2' + 2x_3 + s_1 = 10$$

$$2x_1 + 4x_2' + 3x_3 + s_2 = 15$$

$$x_1, x_2', x_3, s_1, s_2 \geq 0$$

The initial basic feasible solution to this problem is $x_{B_1} = s_1 = 10$, $x_{B_2} = s_2 = 15$. Also for the basic variables s_1 and s_2 no upper bounds are specified, it is therefore assumed that both of these have an upper bound at α. The initial basic feasible solution can be read from the initial simplex table.

		u_j	4	3	3	α	α	
		c_j	3	5	2	0	0	
c_B	x_B	b	x_1	x_2'	x_3	s_1	s_2	$u_i - x_{B_i}$
0	s_1	10	1	2	2	1	0	$\alpha - 10 = \alpha$
0	s_2	15	2	[4]	3	0	1	$\alpha - 15 = \alpha$
$z = 10$		$z_j - c_j$	-3	-5	-2	0	0	

Since $z_2 - c_2 = -5$ is largest negative x_2' will enter into the basis. As none of the basic variables s_1 and s_2 are at their upper bound, thus for deciding about the variable to leave the basis, we compute

$$\theta_1 = \min_i \left\{ \frac{x_{B_i}}{y_{i2}} : y_{i2} > 0 \right\} = \min\left\{ \frac{10}{2}, \frac{15}{4} \right\} = \frac{15}{4}$$

We note that $\theta_1 = \frac{15}{4}$ corresponds to s_2.

$\theta_2 = \alpha$, because all entries in column 2 are positive i.e. $y_{i2} > 0$ for all i

We have $u_2 = 3$.

$$\therefore \theta = \min\{\theta_1, \theta_2, u_2\} = \min\left\{ \frac{15}{4}, \alpha, 3 \right\} = 3$$

We note that $\theta = 3$ corresponds to u_2.

Hence, the non-basic variable x_2' is substituted at the upper bound and remains non-basic. The non-basic variable x_2' at its upper bound can now be put at zero value by using the substitution :

$$x_2' = u_2 - x_2'' = 3 - x_2'', \; 0 \leq x_2'' \leq 3.$$

The value of other basic variables are updated by using

$x_{B_1} = s_1 = x'_{B_1} - y_{12}u_2 = 10 - 2 \times 3 = 4$

$x_{B_2} = s_2 = x'_{B_2} - y_{22}u_2 = 15 - 4 \times 3 = 3$

The data in the previous table can now be updated by putting new value of basic variables and non-basic variable x''_2 as shown in the following table.

		u_i	4	3	3	α	α	
		c_i	3	-5	2	0	0	
c_B	x_B	b	x'_1	x''_2	x_3	s_1	s_2	$u_i - x_{B_i}$
0	s_1	4	1	-2	2	1	0	$\alpha - 4 = \alpha$
0	s_2	3	[2]	-4	3	0	1	$\alpha - 3 = \alpha$
			-3	5	-2	0	0	

In this table the largest negative net evalution is $z_1 - c_1 = -3$. Therefore the variable x_1 will enter into the basis. For deciding about the variable to leave the basis, we compute

$$\theta_1 = \min\left\{\frac{x_{B_i}}{y_{il}} : y_{il} > 0\right\} = \min\left\{\frac{4}{1}, \frac{3}{1}\right\} = 3 \text{ (corresponding to } s_2\text{)}$$

$\theta_2 = \alpha$, because all entries in column 1 are positive i.e. $y_{il} > 0$ for all i.

Here $u_1 = 4$. $\therefore \theta = \min\{\theta_1, \theta_2, u\} = \min\{3, \alpha, 4\}$

$= 3$ (corresponding to θ_1)

Hence, the basic variable s_2 will leave the basis and $y_{21} = 2$ becomes the key element. $\therefore$ x_1 is introduced into the basis and s_2 is removed from the basis. The improved solution is shown in the following table.

		u_i	4	3	3	α	α	
		c_i	3	-5	2	0	0	
c_B	x_B	b	x_1	x''_2	x_3	s_1	s_2	$u_i - x_{B_i}$
0	s_1	5/2	0	0	1/2	1	$-3/2$	$\alpha - 5/2 = \alpha$
3	x_1	3/2	1	[-2]	3/2	0	1/2	$3 - 3/2 = 3/2$
		$z_j - c_j$	0	-1	5/2	0	3/2	

In this table $z_2 - c_2 = -1$ is the only negative value. So x_2'' will enter into the basis. To decide which variable will leave the basis, we compute

$\theta_1 = \alpha$, because all entries in column 2 are non-positive i.e. $y_{i2} \le 0$ for all i.

$$\theta_2 = \min\left\{\frac{u_i - x_{B_i}}{-y_{i2}} : y_{i2} < 0\right\} = \min\left\{\alpha, \frac{5/2}{2}\right\}$$

$$= \frac{5}{4} \text{ (corresponds to } x_1)$$

Here $u_2 = 3$

$$\therefore \theta = \min\{\theta_1, \theta_2, u_2\} = \min\{\alpha, 5/4, 3\}$$

$$= \frac{5}{4} \text{ (corresponds to } \theta_2)$$

Hence, variable x_1 will leave the basis. To put x_1 at its upper bound, we substitute $x_1 = 4 - x_1'$ in the table. The improved solution is shown in the following table.

		u_j	4	3	3	α	α
		c_j	3	−5	2	0	0
c_B	x_B	b	x_1	x_2''	x_3	s_1	s_2
0	s_1	5/2	0	0	1/2	1	−1/2
−5	x_2''	−3/4	−1/2	1	−3/4	0	−1/4
		$z_j - c_j$	−1/2	0	7/4	0	5/4

The value of non-basic variable x_1 at its upper bound 4 can be put at zero level by substituting

$x_1 = 4 - x_1'$; $\quad 0 \le x_1' \le 4$.

The value of other basic variables are updated by using the relationship

$$x_{B_1} = s_1 = x'_{B_1} - y_{11}u_1 = 5/2 - 0 \times 4 = 5/2$$

$$x_{B_2} = x_2'' = x'_{B_2} - y_{21}u_1 = -3/4 - (-1/2) \times 4 = 5/4.$$

The data in the previous table can now be updated by putting new values of basic variables and non-basic variable x_1' as shown in the following table.

c_B	x_B	b	x_1'	x_2''	x_3	s_1	s_2
0	s_1	5/2	0	0	1/2	1	−1/2
−5	x_2''	5/4	1/2	1	−3/4	0	−1/4
$z = 123/4$		$z_j - c_j$	1/2	0	7/4	0	5/4

We note that all $z_j - c_j \geq 0$. So an optimal solution is arrived. The optimal solution is

$x_1' = 0$ or, $4 - x_1 = 0$ or, $x_1 = 4$

$x_2'' = 5/4$ or, $3 - x_2' = 5/4$ or, $3 - (x_2 - 2) = 5/4$ or, $x_2 = 15/4$

and maximum value of $z = 123/4$.

102.13 DECOMPOSITION PRINCIPLE

Some linear programming problems encountered in practice have a large number of variables and/or constraints and so require considerable computing time. However, if the problem has some special structure, it is possible to obtain the solution by applying the decomposition principle developed by Dantzig and wolfe. In the decomposition method, the original problem is decomposed into small subproblems and then these subproblems are solved almost independently. The idea of decomposition, when applicable, has the advantage of making it possible to solve large scale problems which otherwise may be computationally very difficult.

Special Structure of large Scale LPP : We have noted that if the large scale LPP has a special structure then only Decomposition Principle is applicable. The general form of this special structured LPP is as follows :

$$\text{Minimize} \quad f(X) = C_1^T x_1 + C_2^T x_2 + \ldots + C_p^T x_p$$

$$\text{subject to} \quad A_1X_1 + A_2X_2 + \ldots + A_pX_p = b_0$$

$$B_1X_1 = b_1$$

$$B_2X_2 = b_2$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$B_pX_p = b_p$$

$$X_1 \geq 0,\ X_2 \geq 0,\ \ldots\ldots,\ X_p \geq 0$$

where

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m_1} \end{bmatrix}, X_2 = \begin{bmatrix} x_{m_1+1} \\ x_{m_1+2} \\ \vdots \\ x_{m_1+m_2} \end{bmatrix}, \ldots\ldots, X_p = \begin{bmatrix} x_{m_1+m_2} + \cdots + m_{p-1} + 1 \\ x_{m_1+m_2} + \cdots + m_{p-1} + 2 \\ \vdots \\ x_{m_1+m_2} + \cdots + m_{p-1} + m_p \end{bmatrix}$$

and $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$.

Let the size of the matrix A_k be $r_o \times m_k$ and that of B_k be $r_k \times m_k$. So the number of variables in this LPP is $m_1 + m_2 + ... + m_p$ and number of constraints is $r_0 + r_1 + r_2 + ... + r_p$.

Here the first set $A_1X_1 + A_2X_2 + ... + A_pX_p = b_0$ is the set of common constraints and the remaining sets i.e. $B_1X_1 = b_1, B_2X_2 = b_2, ..., B_pX_p = b_p$ are p sets of independent constraints.

Solution Procedure : The basic solution procedure using decomposition principle is given by the following steps.

Step 1: For each k = 1, 2, ..., p find the extreme points of the convex set of feasible solutions of

$$B_kX_k = b_k$$

$$X_k \geq 0$$

Let the number of extreme points be s_k and the extreme points be $X_1^{(k)}, X_2^{(k)}, ..., X_{s_k}^{(k)}$. Then any solution of this system can be represented as the following convex combination

$$X_k = \mu_{k1}X_1^{(k)} + \mu_{k2}X_2^{(k)} + ... + \mu_{ks_k}X_{s_k}^{(k)} \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

$$\mu_{k1} + \mu_{k2} + ... + \mu_{ks_k} = 1$$

$$0 \leq \mu_{ki} \leq 1, \quad i = 1, 2, ..., s_k$$

Thus new equations (2) gives the complete solution space of the system (1).

Step 2 : Use (2) in the given LPP to obtain the following LPP equivalent to the given LPP.

$$\text{Minimize} \quad f(X) = C_1^T\left(\sum_{i=1}^{s_1}\mu_{1i}X_i^{(1)}\right) + C_2^T\left(\sum_{i=1}^{s_2}\mu_{2i}X_i^{(2)}\right) + ... + C_p^T\left(\sum_{i=1}^{s_p}\mu_{pi}X_i^{(p)}\right)$$

subject to $$A_1\left(\sum_{i=1}^{s_1}\mu_{1i}X_i^{(1)}\right)+A_2\left(\sum_{i=1}^{s_2}\mu_{2i}X_i^{(2)}\right)+\ \ldots\ +A_p\left(\sum_{i=1}^{s_p}\mu_{pi}X_i^{(p)}\right)=b_0$$

$$\sum_{i=1}^{s_1}\mu_{1i} = 1$$

$$\sum_{i=1}^{s_2}\mu_{2i} = 1$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$\sum_{i=1}^{s_p}\mu_{pi} = 1$$

$$\mu_{ji} \geq 0,\ i = 1, 2, \ldots, s_j;\ j = 1, 2, \ldots, p$$

Since $X_1^{(k)}, X_2^{(k)}, \ldots, X_{s_k}^{(k)}, C_k, A_k$ are known for all $k = 1, 2, \ldots, p$ the above problem is a LPP with decision variables as $\mu_{ji}, i = 1, 2, \ldots, s_j;\ j = 1, 2, \ldots, p$.

Step 3 : Solve the equivalent LPP obtained in Step 2 by any standard method. Let the optimal solution be $\mu_{ji} = \mu_{ji}^*, i = 1, 2, \ldots, s_j;\ j = 1, 2, \ldots, p$.

Find the optimal solution of the original LPP as

$$X^* = \begin{bmatrix} X_1^* \\ X_2^* \\ \vdots \\ X_p^* \end{bmatrix} \qquad \text{where } X_k^* = \sum_{i=1}^{s_k}\mu_{kj}^* X_i^{(k)},\ k = 1, 2, \ldots, p.$$

Note :- It is to be noted that the new equivalent LPP has only $r_0 + p$ constraints whereas the original LPP has $r_0 + \sum_{k=1}^{p} r_k$ constraints.

102.14. Illustrative examples :

Example 102.14.1. : Find the equivalent LPP using decomposition principle for the following LPP

Maximize $f = x_1 + 2x_2 + 2x_3 + 3x_4$ .....(1)

subject to $x_1 + x_2 + x_3 + x_4 \leq 1000$

$$x_1 + x_3 \leq 500$$
$$x_1 + x_2 \leq 600$$
$$x_1 - 2x_2 \leq 0$$
$$-2x_3 + x_4 \leq 0$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

Solution : The given LPP can be restated in the matrix notation as follows :

$$\text{Maximize} \quad f(X) = C_1^T X_1 + C_2^T X_2 \qquad \text{.....(2)}$$

$$\text{Subject to} \quad A_1X_1 \quad A_2X_2 \leq b_0$$
$$B_1X_1 \leq b_1$$
$$B_2X_2 \leq b_2$$
$$X_1 \geq 0,\ X_2 \geq 0.$$

where $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $X_2 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$, $C_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $C_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},\ b_0 = \begin{bmatrix} 1000 \\ 500 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix},\ b_1 = \begin{bmatrix} 600 \\ 0 \end{bmatrix},\ B_2 = [-2 \quad 1]$$

$b_2 = 0$ and $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$.

The first subsidiary constraint set is

$$B_1X_1 \leq b_1,\ X_1 \geq 0$$

i.e. $x_1 + x_2 \leq 600$

$x_1 - 2x_2 \leq 0$

$x_1, x_2 \geq 0$

The feasible region is shown in Fig 1 and is the region OAB.

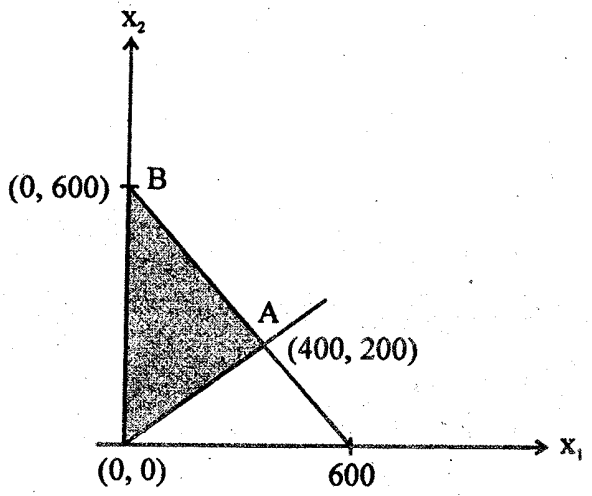


Fig 1

The extreme points of this convex feasible region are 0 (0, 0), A (400, 200) and B (0, 600)

Thus we have

$$X_1^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ X_2^{(1)} = \begin{bmatrix} 400 \\ 200 \end{bmatrix},\ \text{and } X_3^{(1)} = \begin{bmatrix} 0 \\ 600 \end{bmatrix}$$

$\therefore$ Any point in the convex feasible region OAB is given by

$$X_1 = \mu_{11}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \mu_{12}\begin{bmatrix} 400 \\ 200 \end{bmatrix} + \mu_{13}\begin{bmatrix} 0 \\ 600 \end{bmatrix} = \begin{bmatrix} 400\mu_{12} \\ 200\mu_{12} + 600\mu_{13} \end{bmatrix} \quad \text{.....(3)}$$

with $\mu_{11} + \mu_{12} + \mu_{13} = 1$

$0 \le \mu_{1i} \le 1,\ i = 1, 2, 3.$

The second subsidiary constraint set is

$$B_2X_2 \le b,\ X_2 \ge 0$$

i.e. $-2x_3 + x_4 \le 0$

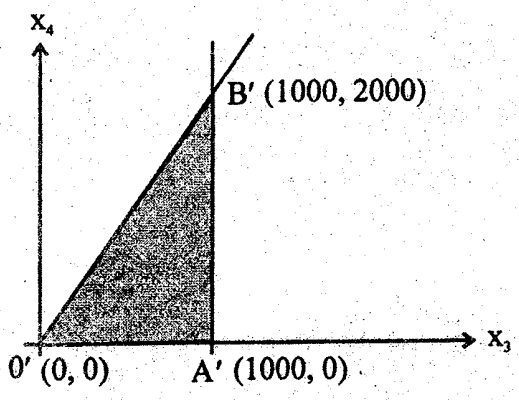
$x_3,\ x_4 \ge 0$

We note that the feasible region given by these constraints is unbounded.

In order to make the feasible region bounded, we add the constrant $x_3 \le 1000$. Thus we have

$$-2x_3 + x_4 \leq 0$$
$$x_3 \leq 1000$$
$$x_3, x_4 \geq 0$$

The feasible region is shown in Fig 2 and is the region O'A'B'



Thus we have

$$X_1^{(2)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, X_2^{(2)} = \begin{bmatrix} 1000 \\ 0 \end{bmatrix} \text{ and } X_3^{(2)} = \begin{bmatrix} 1000 \\ 2000 \end{bmatrix}$$

$\therefore$ Any point in the convex feasible region O'A'B' is given by

$$X_2 = \mu_{21}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \mu_{22}\begin{bmatrix} 1000 \\ 0 \end{bmatrix} + \mu_{23}\begin{bmatrix} 1000 \\ 2000 \end{bmatrix} = \begin{bmatrix} 1000\,\mu_{22} + 1000\,\mu_{23} \\ 2000\,\mu_{23} \end{bmatrix}$$

with $\mu_{21} + \mu_{22} + \mu_{23} = 1$ ..... (4)

$0 \leq m_{2i} \leq 1,\ i = 1, 2, 3.$

From (3) and (4) we have

$x_1 = 400\mu_{12}$ ..... (5)

$x_2 = 200\mu_{12} + 600\mu_{13}$ ..... (6)

$x_3 = 1000\mu_{22} + 1000\mu_{23}$ ..... (7)

$x_4 = 2000\mu_{23}$ ..... (8)

Putting the values of x_1, x_2, x_3, x_4 in the LPP (1) we get the following equivalent LPP

Maximize $f = 400\mu_{12} + 2(200\mu_{12} + 600\mu_{13}) + 2(1000\mu_{22} + 1000\mu_{23}) + 3(2000\mu_{23})$

Subject to $400\mu_{12} + (200\mu_{12} + 600\mu_{13}) + (1000\mu_{22} + 1000\mu_{23}) + 2000\mu_{23} \leq 1000$

$$400\mu_{12} + (1000\mu_{22} + 1000\mu_{23}) \le 500$$

$$\mu_{11} + \mu_{12} + \mu_{13} = 1$$

$$\mu_{21} + \mu_{22} + \mu_{23} = 1$$

$$\mu_{11} \ge 0,\ \mu_{12} \ge 0,\ \mu_{13} \ge 0,\ \mu_{21} \ge 0,\ \mu_{22} \ge 0,\ \mu_{23} \ge 0$$

i.e. Maximize $f = 800\mu_{12} + 1200\mu_{13} + 2000\mu_{22} + 8000\mu_{23}$

subject to $600\mu_{12} + 600\mu_{13} + 1000\mu_{22} + 3000\mu_{23} \le 1000$

$400\mu_{12} + 1000\mu_{22} + 1000\mu_{23} \le 500$

$\mu_{11} + \mu_{12} + \mu_{13} = 1$

$\mu_{21} + \mu_{22} + \mu_{23} = 1$

and $\mu_{11}, \mu_{12}, \mu_{13}, \mu_{21}, \mu_{22}, \mu_{23} \ge 0.$

This LPP is an equivalent LPP to the given LPP and can be solved by simplex method. After getting this optimal solution the optimal values of the decision variables are obtained using (5), (6), (7) and (8).

102.15 SUMMARY

Dual Simplex method is found to be very useful in a large class of LPP. It is simple to handle and size of the tables are not large as no artificial variables are introduced. The theorems behind the development of the method are proved and the method is illustrated through examples. The method is then modified to handle more LPP. To tackle LPP with bounded variables, method of bounded variables is discussed and explained by examples.

102.16 Self Assesment Questions.

1. Use dual simplex method to solve the LPP

   Maximize $z = -2x_1 - 3x_2 - x_3$

   subject to $2x_1 + x_2 + 2x_3 \ge 3$

   $3x_1 + 2x_2 + x_3 \ge 4$

   $x_1, x_2, x_3 \ge 0$

   [Ans : $x_1 = 5/4,\ x_2 = 0,\ x_3 = 1/4,\ z_{max} = -11/4$]

2. Use dual simplex method to solve the LPP

   Minimize $z = 6x_1 + 7x_2 + 3x_3 + 5x_4$

   subject to $5x_1 + 6x_2 - 3x_3 + 4x_4 \ge 12$

   $x_2 + 5x_3 - 6x_4 \ge 10$

$$2x_1 + 5x_2 + x_3 + x_4 \geq 8$$

$$x_1, x_2, x_3, x_4 \geq 0$$

[Ans : $x_1 = 0$, $x_2 = 30/11$, $x_3 = 16/11$, $x_4 = 0$, $z_{min} = 258/11$]

3. Use dual simplex method to solve the LPP

Minimize $z = 10x_1 + 6x_2 + 2x_3$

subject to $-x_1 + x_2 + x_3 \geq 1$

$3x_1 + x_2 - x_3 \geq 2$

$x_1, x_2, x_3 \geq 0$

[Ans : $x_1 = 1/4$, $x_2 = 5/4$, $x_3 = 0$, $z_{min} = 10$]

4. Solve by dual simplex method the following LPP

Minimize $z = 6x_1 + x_2$

subject to $2x_1 + x_2 \geq 3$

$x_1 - x_2 \geq 0$

$x_1, x_2 \geq 0$

[Ans : $x_1 = 1$, $x_2 = 1$, $z_{min} = 7$]

5. Solve by dual simplex method

Maximize $z = -2x_1 - x_3$

subject to $x_1 + x_2 - x_3 \geq 5$

$x_1 - 2x_2 + 4x_3 \geq 8$

$x_1, x_2, x_3 \geq 0$

[Ans : $x_1 = 0$, $x_2 = 14$, $x_3 = 9$, $z_{max} = -9$]

6. Solve by following LPP by dual simplex method

Maximize $z = -3x_1 - 2x_2$

subject to $x_1 + x_2 \geq 1$

$x_1 + x_2 \leq 7$

$x_1 + 2x_2 \geq 10$

$x_2 \leq 3$

$x_1, x_2 \geq 0$

[Ans : $x_1 = 4$, $x_2 = 3$, $z_{max} = -18$]

7. Solve by dual simplex method

Minimize $z = 2x_1 + 3x_2$

subject to $2x_1 + 3x_2 \leq 30$

$x_1 + 2x_2 \geq 10$

$x_1 - x_2 \geq 0$

$x_1 \geq 5$

$x_2 \geq 0$

[Ans : $x_1 = 5,\ x_2 = 5/2,\ z_{min} = 35/2$]

8. Solve the following LPP by dual simplex method

Minimize $z = x_1 + x_2$

subject to $2x_1 + x_2 \geq 2$

$-x_1 - x_2 \geq 1$

$x_1,\ x_2 \geq 0$

[Ans : No feasible solution]

9. Solve the following LPP by dual simplex method

Minimize $z = x_1 + 2x_2 + 3x_3$

subject to $2x_1 - x_2 + x_3 \geq 4$

$x_1 + x_2 + 2x_3 \leq 8$

$x_2 - x_3 \geq 2$

$x_1,\ x_2,\ x_3 \geq 0$

[Ans : $x_1 = 3,\ x_2 = 2,\ x_3 = 0,\ z_{min} = 7$]

10. Solve by dual simplex method

Minimize $z = -2x_1 - x_2 - x_3$

subject to $4x_1 + 6x_2 + 3x_3 \leq 8$

$x_1 - 9x_2 + x_3 \leq -3$

$2x_1 + 3x_2 - 5x_3 \geq 4$

$x_1,\ x_2,\ x_3 \geq 0$

[Ans : $x_1 = 9/7,\ x_2 = 10/21,\ x_3 = 0,\ z_{min} = -64/21$]

11. Solve by dual simplex method

Minimize $z = 3x_1 + 2x_2 + x_3 + 4x_4$

subject to $2x_1 + 4x_2 + 5x_3 + x_4 \geq 10$

$3x_1 - x_2 + 7x_3 - 2x_4 \geq 2$

$5x_1 + 2x_2 + x_3 + 6x_4 \geq 15$

$x_1, x_2, x_3, x_4 \geq 0$

[Ans : $x_1 = 65/23$, $x_2 = 0$, $x_3 = 20/23$, $x_4 = 0$, $z_{min} = 215/23$]

12. Solve the following LPP by dual simplex method

Minimize $z = 4x_1 + 3x_2 + x_3 + 2x_4$

subject to $3x_1 + 4x_2 + 5x_3 + 2x_4 \geq 10$

$2x_1 - x_2 + 6x_3 - x_4 \geq 3$

$6x_1 + 3x_2 + x_3 + 4x_4 \geq 18$

$x_1, x_2, x_3, x_4 \geq 0$

[Ans : $x_1 = 0$, $x_2 = 0$, $x_3 = 6/5$, $x_4 = 21/5$, $z_{min} = 48/5$]

13. Using artificial constraint procedure, solve the following problem by dual simplex method and show that the problem has no feasible solution

Minimize $z = -x_1 + x_2$

subject to $x_1 - 4x_2 \geq 5$

$x_1 - 3x_2 \leq 1$

$2x_1 - 5x_2 \geq 1$

$x_1, x_2 \geq 0$

14. Use the artificial constraint method to find the initial basic solution of the following problem and then apply the dual simplex algorithm to solve it

Maximize $z = x_1 - 3x_2 - 2x_3$

subject to $x_2 - 2x_3 \geq 2$

$x_1 - 4x_2 - 6x_3 = 8$

$2x_2 + x_3 \leq 5$

$x_1, x_2, x_3 \geq 0$

[Ans : $x_1 = 94/5$, $x_2 = 12/5$, $x_3 = 1/5$, $z_{max} = 56/5$]

102.17. Suggested Further Readings

1. Linear Programming and Game Theory : Chakravorty & Ghosh; Moulik Library.
2. Operations Research : Kanti Swarup, P. K. Gupta, Man Mohan; Sultan Chand & Sons.
3. Operations Research : J. K. Sharma; Macmillan India Limited.
4. Linear Programming and Theory of Games : S. D. Sharma and Hemlata Sharma; Keder Nath Ram Nath & Co.
5. Operations Research : R. K. Gupta; Krishna Prakashan Mandir.
6. Linear Programming and Theory of Games : P. K. Gupta; Khanna Publishers.

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

Part - II

Paper - IX (Special paper)

OPERATIONS RESEARCH

Module No. 103

(Post Optimality Analysis)

STRUCTURE :

103.1 Introduction

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103.3 Discrete changes in the cost vector

103.4 Illustrative examples

103.5 Discrete changes in the requirement vector

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103.7 Addition of a single variable

103.8 Illustrative examples

103.9 Deletion of a variable

103.10 Illustrative examples

103.11 Addition of a new constraint

103.12 Illustrative examples

103.13 Summary

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POST OPTIMALITY ANALYSIS

103.1 INTRODUCTION

Often an error is discovered in the data after the attainment of an optimal solution to the problem. In such a situation there are two alternatives, either to solve the problem from begining or to device some method to use the optimal table. Undoubtedly the second one will save time and space and is named as post optimality analysis.

Also in practical situation the values of the co-efficient matrix A, the components of the requirement

vector and the cost vector are neither known exactly nor they are constant for all time and for all situations. So it is important to know how sensitive the optimal solution is to small changes in these parameters. By sensitiveness we mean fulfilments of the condition of optimality as well as determining the limits of variations of these parameters for the solution to remain optimal.

We shall study the following effects of changes in the

i) coefficients c_j of the objective function

ii) components of the requirement vector b

iii) addition of a new variable

iv) deletion of a variable

v) addition of a new constraint

103.2 OBJECTIVE

Post optimality analysis is very useful to tackle the problems in real life. In reality the problems occurring are in general large in size and often an error is discovered in the data after getting the solution. In such a situation to solve the problem a fresh is very cumbersome and time-consuming. Post optimality analysis is the only rescue where the final optimal solution obtained for the original problem is used very efficiently to get the optimal solution of the changed problem.

103.3 DISCRETE CHANGES IN THE COST VECTOR

Let x_B be the optimal basic solution of the LPP

Maximize $z = cx$

subject to $Ax = b$

$x \geq 0$

where c, $x^T \in R^n$, $b^T \in R^m$ and A is an m×n real matrix. let Δc_k be the amount by which c_k is changed. So the new value of c_k is $c_k^* = c_k + \Delta c_k$.

We know that $x_B = B^{-1} b$ and so it is independent of c. As initially x_B was BFS it will remain so after the change. The optimality condition is $z_j - c_j \geq 0$ for all j i.e. $\left[c_B B^{-1}\ I\right]\begin{bmatrix} A \\ -c \end{bmatrix} \geq 0$. It involves c. So change in c will affect this condition. Thus when c_k is changed to c_k^*, the solution x_B may or may not remain optimal solution though it remains BFS.

Two cases will arise

i) c_k is not in c_B

ii) c_k is in c_B

Case (i). Here c_k is not in c_B. The net evaluations are the components of $c_B B^{-1}A - C$ and as $x_B = B^{-1}b$ was optimal solution we have $c_B B^{-1}A - c \geq 0$ i.e., $z_j - c_j \geq 0 \forall j$.

We note that when c_k is changed to c_k^* only k th component of net evaluation will change.

Thus for all j = 1, 2, ..., k-1, k+1, ...n i.e., for all $j \neq k$.

we have new net evaluations

$$z_j^* - c_j^* = z_j - c_j \geq 0 \quad [\text{as } z_j - c_j \geq 0 \text{ for all } j]$$

& for $j = k$ we have $z_k^* - c_k^* = z_k - (c_k + \Delta c_k)$

$$= (z_k - c_k) - \Delta c_k$$

We have $z_k - c_k \geq 0$. Therefor for all Δc_k, $z_k^* - c_k^*$ will not remain non negative.

Thus x_B will remain optimal solution for the changed LPP if $z_k - c_k - \Delta c_k \geq 0$ i.e., if $\Delta c_k \leq z_k - c_k$.

Case (ii). Here c_k is one component of c_B. Let $c_k = c_{B\lambda}$ and so x_k is a basic variable. Thus y_k is a unit vector with its λth component as 1.

The new value of $z_k - c_k$ is given by

$$z_k^* - c_k^* = \sum_{\substack{i=1 \\ i \neq \lambda}}^{m} c_{B_i} y_{ik} + c_k^* .1 - c_k^* = 0 \quad [\because y_{ik} = 0 \forall i \neq \lambda]$$

For $j \neq k$, new value of $z_j - c_j$ is given by

$$z_j^* - c_j^* = \left(\sum_{\substack{i=1 \\ i \neq \lambda}}^{m} c_{B_i} y_{ij} + c_k^* y_{\lambda j} \right) - c_j^*$$

$$= \sum_{i \neq \lambda} c_{B_i} y_{ij} + (c_k + \Delta c_k) y_{\lambda j} - c_j \quad [\because c_j^* = c_j \ \ \forall j \neq k]$$

$$= \sum_{i \neq \lambda} c_{B_i} y_{ij} + c_{B\lambda} y_{\lambda j} + \Delta c_k y_{\lambda j} - c_j \quad [\because c_k = c_{B\lambda}]$$

$$= \sum_{i=1}^{m} c_{B_i} y_{ij} + \Delta c_k y_{\lambda j} - c_j$$

$$= z_j - c_j + \Delta c_k y_{\lambda j} \quad \left[\because z_j = \sum_{i=1}^{m} c_{B_i} y_{ij}\right]$$

$\therefore$ x_B remains optimal solution

if $z_j - c_j + \Delta c_k y_{\lambda j} \geq 0 \quad \forall j \neq k$

i.e., if $\Delta c_k y_{\lambda j} \geq -(z_j - c_j) \quad \forall j \neq k$

Now for $y_{\lambda j} = 0$ this condition is fulfilled automatically.

as $z_j - c_j \geq 0$.

For $y_{\lambda j} > 0$ this condition is satisfied

$$\text{if } \Delta c_k \geq -\frac{z_j - c_j}{y_{\lambda j}} \quad \forall j \neq k$$

$$\text{i.e., if } -\frac{z_j - c_j}{y_{\lambda j}} \leq \Delta c_k \quad \forall j \neq k.$$

$$\therefore \text{We must have } \max_{j \neq k} y_{\lambda j} > 0 \left\{-\frac{z_j - c_j}{y_{\lambda j}}\right\} \leq \Delta c_k$$

For $y_{\lambda j} < 0$ this condition is satisfied

$$\text{if } \Delta c_k \leq -\frac{z_j - c_j}{y_{\lambda j}} \forall \ j \neq k$$

$$\therefore \text{We must have } \Delta c_k \leq \min_{j \neq k} y_{\lambda j} < 0 \left\{-\frac{z_j - c_j}{y_{\lambda j}}\right\}$$

These two conditions can be combined as

$$\max_{j \neq k} y_{\lambda j} > 0 \left\{-\frac{z_j - c_j}{y_{\lambda j}}\right\} \leq \Delta c_k \leq \min_{j \neq k} y_{\lambda j} < 0 \left\{-\frac{z_j - c_j}{y_{\lambda j}}\right\}$$

Hence if Δc_k lies in this range then the solution x_B remains optimal and if Δc_k falls outside this range then at least one $z_j - c_j$ will be negative and the solution will no longer remain optimal.

If no $y_{\lambda j} > 0$, then there is no lower bound of Δc_k and if no $y_{\lambda j} < 0$, then there is no upper bound of Δc_k.

103.4 ILLUSTRATIVE EXAMPLES

Example 103.3.1 : For the LPP

$$\text{Maximize} \quad z = 3x_1 + 5x_2$$

$$\text{subject to} \quad x_1 + x_2 \le 1$$

$$2x_1 + 3x_2 \le 1$$

$$x_1, x_2 \ge 0$$

the optimal table for the simplex method is found to be

c_B	y_B	x_B	y_1	y_2	y_3	y_4
0	y_3	2/3	1/3	0	1	−1/3
5	y_2	1/3	2/3	1	0	1/3
		5/3	1/3	0	0	5/3

Obtain the ranges of c_j for which the optimal solution remains optimal when changes one at a time.

Solution : The optimal table is

		c_j	3	5	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4
0	y_3	2/3	1/3	0	1	−1/3
5	y_2	1/3	2/3	1	0	1/3
		5/3	1/3	0	0	5/3

From this table we see that $c_2 \in c_B$ and $c_1 \notin c_B$.

For variation in c_1 : In this case as $c_1 \notin c_B$ we use the

formula $\quad \Delta c_j \le z_j - c_j$

i.e. $\quad \Delta c_1 \le 1/3$

i.e. $\quad -\alpha < \Delta c_1 \le 1/3$

The range of c_1^* is given by

$$-\alpha + c_1 < c_1 + \Delta c_1 \le c_1 + 1/3$$

or, $$-\alpha + 3 < c_1^* \le 3 + 1/3$$

or, $$-\alpha < c_1^* < 3\frac{1}{3}.$$

For variation of c_2 : In this case $c_2 \in c_B$ and $c_2 = c_{B_2}$. The range of Δc_2 is given by

$$\max_{\substack{y_{2j} > 0 \\ j \neq 2}} \left\{ -\frac{z_j - c_j}{y_{2j}} \right\} \le \Delta c_2 \le \min_{\substack{y_{2j} < 0 \\ j \neq 2}} \left\{ -\frac{z_j - c_j}{y_{2j}} \right\}$$

or, $$\max\left\{ -\frac{1/3}{2/3}, -\frac{5/3}{1/3} \right\} \le \Delta c_2 < \alpha \quad [\because \text{ there is no } y_{2j} < 0]$$

or, $$\max\{-1/2, -5\} \le \Delta c_2 < \alpha$$

or, $$-1/2 \le \Delta c_2 < \alpha$$

or, $$c_2 - 1/2 < c_2 + \Delta c_2 < c_2 + \alpha$$

or, $$5 - 1/2 < c_2^* < 5 + \alpha$$

or, $$9/2 < c_2^* < \alpha$$

Example 103.3.2 : The optimal solution of theLPP

$$\begin{aligned} \text{Maximize} \quad & z = 6x_1 - 2x_2 + 3x_3 \\ \text{subject to} \quad & 2x_1 - x_2 + 2x_3 \le 2 \\ & x_1 \quad\quad + 4x_3 \le 4 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

is contained in the table

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
6	y_1	4	1	0	4	0	1
−2	y_2	6	0	1	6	−1	2
$z_j - c_j$		$z = 12$	0	0	9	2	2

Find the ranges of the cost components when (i) changed one at a time (ii) changed two at a time (iii) changed all three at a time to keep the optimal solution same.

Solution :

(i) When one component is changed at a time :

For change of $c_1 = 6$ to c_1^* we have the corresponding changed table as

		c_j	c_1^*	-2	3	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
c_1^*	y_1	4	1	0	4	0	1
-2	y_2	6	0	1	6	-1	2
			0	0	$4c_1^* - 12$	2	$c_1^* - 4$

This table becomes optimal table

if $4c_1^* - 12 \geq 0$ and $c_1^* - 4 \geq 0$

i.e. if $c_1^* \geq 3$ and $c_1^* \geq 4$

i.e. if $c_1^* \geq 4$.

For change of $c_2 = -2$ to c_2^* the table corresponding to the final table becomes

		6	c_2^*	3	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
6	y_1	4	1	0	4	0	1
c_2^*	y_2	6	0	1	6	-1	2
			0	0	$24 + 6c_2^*$	$-c_2^*$	$6 + 2c_2^*$

This table becomes the optimal table

if $24 + 6c_2^* \geq 0$ and $-c_2^* \geq 0$ and $6 + 2c_2^* \geq 0$

i.e. if $c_2^* \geq -4$ and $c_2^* \leq 0$ and $c_2^* \geq -3$

i.e. if $-3 \leq c_2^* \leq 0$.

For change of c_3=3 to c_3^* the modified table is

			6	–2	c_3^*	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
6	y_1	4	1	0	4	0	1
–2	y_2	6	0	1	6	– 1	2
			0	0	$12 - c_3^*$	2	2

This table remains optimal table

if $12 - c_3^* \geq 0$

i.e. if $c_3^* \leq 12$

ii) When two components are changed at a time.

For the change of c_1=6 and c_2 = –2 to c_1^* and c_2^* the modified table is

			c_1^*	c_2^*	3	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
c_1^*	y_1	4	1	0	4	0	1
c_2^*	y_2	6	0	1	6	– 1	2
			0	0	$4c_1^* + 6c_2^* - 3$	$-c_2^*$	$c_1^* + 2c_2^*$

This table becomes optimal table if all $z_j - c_j \geq 0$

i.e. if $4c_1^* + 6c_2^* - 3 \geq 0$ and $-c_2^* \geq 0$ and $c_1^* + 2c_2^* \geq 0$

i.e. if $c_2^* \geq \frac{3 - 4c_1^*}{6}$ and $c_2^* \leq 0$ and $c_2^* \geq -\frac{c_1^*}{2}$

i.e. if $\max\left\{\frac{3 - 4c_1^*}{6}, -\frac{c_1^*}{2}\right\} \leq c_2^* \leq 0$ and c_1^* any real number.

For the change of c_1 = 6 and c_3=3 to c_1^* and c_3^* respectively the modified table is

			c_1^*	-2	c_3^*	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
c_1^*	y_1	4	1	0	4	0	1
-2	y_2	6	0	1	6	-1	2
			0	0	$4c_1^* - 12 - c_3^*$	2	$c_1^* - 4$

This table remains optimal table if all $z_j - c_j \geq 0$

i.e. if $4c_1^* - 12 - c_3^* \geq 0$ and $c_1^* - 4 \geq 0$

i.e. if $c_1^* \geq \frac{12 + c_3^*}{4}$ and $c_1^* \geq 4$

i.e. if $c_1^* \geq \max\left\{4,\ 3 + \frac{c_3^*}{4}\right\}$ and c_3^* any real number.

For the change of $c_2 = -2$ and $c_3 = 3$ to c_2^* and c_3^* respectively the modified table is

			6	c_2^*	c_3^*	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
6	y_1	4	1	0	4	0	1
c_2^*	y_2	6	0	1	6	-1	2
			0	0	$24 + 6c_2^* - c_3^*$	$-c_2^*$	$6 + 2c_2^*$

This table remains optimal table if all $z_j - c_j \geq 0$

i.e. if $24 + 6c_2^* - c_3^* \geq 0$ and $-c_2^* \geq 0$ and $6 + 2c_2^* \geq 0$

i.e. if $c_2^* \geq \frac{c_3^* - 24}{6}$ and $c_2^* \leq 0$ and $c_2^* \geq -3$

i.e. if $\max\left\{\frac{c_3^* - 24}{6}, -3\right\} \leq c_2^* \leq 0$ and c_3^* any real number.

iii) When all the three components are changed together :

Let $c_1 = 6, c = -2, c_3 = 3$ be changed respectively to c_1^*, c_2^*, c_3^*. The modified table obtained from old optimal table is

			c_1^*	c_2^*	c_3^*	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5
c_1^*	y_1	4	1	0	4	0	1
c_2^*	y_2	6	0	1	6	-1	2
			0	0	$4c_1^* + 6c_2^* - c_3^*$	$-c_2^*$	$c_1^* + 2c_2^*$

This table becomes an optimal table if all $z_j - c_j \geq 0$

i.e. if $4c_1^* + 6c_2^* - c_3^* \geq 0$ and $-c_2^* \geq 0$ and $c_1^* + 2c_2^* \geq 0$

i.e. if $c_2^* \geq \frac{c_3^* - 4c_1^*}{6}$ and $c_2^* \leq 0$ and $c_2^* \geq -\frac{c_1^*}{2}$

i.e. if $\max\left\{\frac{c_3^* - 4c_1^*}{6}, -\frac{c_1^*}{2}\right\} \leq c_2^* \leq 0$ and c_1^* any real number

and c_3^* any real number.

103.4 DISCRETE CHANGE IN THE REQUIREMENT VECTOR

Let x_B be the optimal BFS of the LPP

Maximize $z = cx$

subject to $Ax = b,\ x \geq 0$

Where $c,\ x^T \in R^n,\ b^T \in R^m$ and A is an m×n real matrix. We have $x_B = B^{-1}b$ and the net evaluations are the components $\begin{bmatrix} c_B B^{-1} & 1 \end{bmatrix}\begin{bmatrix} A \\ -c \end{bmatrix}$. From these we see that x_B depends on b but net evaluations are independent of b. So change made in b will not affect optimality conditions i.e., optimal solution will remain optimal but it will change the solution x_B and it may become negative i.e., infeasible.

Let the component b_k of b be changed to $b_k^* = b_k + \Delta b_k$. So the old solution $x_B = B^{-1}b$ becomes

$x_B^* = B^{-1}b^*$ where $b^* = [b_1, b_2, \ldots, b_{k-1}, b_k + \Delta b_k, b_{k+1}, \ldots, b_m]^T$.

Thus

let $B^{-1} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \cdots & \cdots & & \cdots \\ \cdots & \cdots & & \cdots \\ b_{m1} & b_{m2} & & b_{mm} \end{bmatrix}$

$$\therefore \quad x_B^* = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \cdots & \cdots & & \cdots \\ \cdots & \cdots & & \cdots \\ b_{m1} & b_{m2} & & b_{mm} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{k-1} \\ b_k + \Delta b_k \\ b_{k+1} \\ \vdots \\ b_m \end{bmatrix}$$

$$= \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \cdots & \cdots & & \cdots \\ \cdots & \cdots & & \cdots \\ \cdots & \cdots & & \cdots \\ b_{m1} & b_{m2} & & b_{mm} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{k-1} \\ b_k \\ b_{k+1} \\ \vdots \\ b_m \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2k} & \cdots & b_{2m} \\ \cdots & \cdots & & \cdots & \cdots & \\ \cdots & \cdots & & \cdots & \cdots & \\ \cdots & \cdots & & \cdots & \cdots & \\ b_{m1} & b_{m2} & \cdots & b_{mk} & \cdots & b_{mm} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \Delta b_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= B^{-1}b + \begin{bmatrix} b_{1k} & \Delta b_k \\ b_{2k} & \Delta b_k \\ \cdots & \\ \cdots & \\ \cdots & \\ b_{mk} & \Delta b_k \end{bmatrix}$$

$$\begin{bmatrix} x^*_{B_1} \\ x^*_{B_2} \\ \vdots \\ \vdots \\ x^*_{B_m} \end{bmatrix} = \begin{bmatrix} x_{B_1} \\ x_{B_2} \\ \vdots \\ \vdots \\ x_{B_m} \end{bmatrix} + \begin{bmatrix} b_{1k} & \Delta b_k \\ b_{2k} & \Delta b_k \\ \vdots & \\ \vdots & \\ b_{mk} & \Delta b_k \end{bmatrix}$$

$\therefore\ x^*_{B_i} = x_{B_i} + b_{ik}\Delta b_k$ for all i = 1, 2, ..., m.

As we have noted this solution is optimal or better than optimal but may not be feasible though basic.

Thus x^*_B will be an optimal BFS

if $x^*_{B_i} \geq 0$ for all i = 1, 2, ..., m

i.e. if $x_{B_i} + b_{ik}\Delta b_k \geq 0$ for all i = 1, 2, ..., m

i.e. if $b_{ik}\Delta b_k \geq -x_{B_i}$ for all i = 1, 2, ..., m

For all $b_{ik} = 0$ this condition is satisfied.

For all $b_{ik} > 0$ this condition is satisfied if $\Delta b_k \geq -\dfrac{x_{B_i}}{b_{ik}}$

$\therefore$ We need $-\dfrac{x_{B_i}}{b_{ik}} \leq \Delta b_k$ for all $b_{ik} > 0$

i.e. we need $\max\left\{-\dfrac{x_{B_i}}{b_{ik}} : b_{ik} > 0\right\} \leq \Delta b_k$

Again for all $b_{ik} < 0$ this condition is satisfied if $\Delta b_k \leq -\dfrac{x_{B_i}}{b_{ik}}$

$\therefore$ We need $\Delta b_k \leq -\dfrac{x_{B_i}}{b_{ik}}$ for all $b_{ik} < 0$

i.e., we need $\Delta b_k \leq \min\left\{-\dfrac{x_{B_i}}{b_{ik}} : b_{ik} < 0\right\}$

Hence x^*_B will be optimal basic feasible solution if Δb_k is selected satisfying the condition

$$\max\left\{-\frac{x_{Bi}}{b_{ik}}: b_{ik} > 0\right\} \le \Delta b_k \le \min\left\{-\frac{x_{Bi}}{b_{ik}}: b_{ik} < 0\right\}$$

103.5 ILLUSTRATIVE EXAMPLES

Example 103.5.1 : Given the LPP

$$\text{Maximize} \quad z = -x_1 + 2x_2 - x_3$$

$$\text{subject to} \quad 3x_1 + x_2 - x_3 \le 10$$

$$-x_1 + 4x_2 + x_3 \ge 6$$

$$x_2 + x_3 \le 4$$

$$x_1, x_2, x_3 \ge 0$$

Determine the ranges for discrete changes of the components of b when changed one at a time, so as to maintain the optimality of the current optimum solution for the LPP.

Solution : Introducing slack variables $x_4 \ge 0$, $x_6 \ge 0$, surplus variable $x_5 \ge 0$ and artificial variable $x_7 \ge 0$ we have the standard form as follows

$$\text{Maximize} \quad z = -x_1 + 2x_2 - x_3 + ox_4 + ox_5 + ox_6 - Mx_7$$

$$\text{subject to} \quad 3x_1 + x_2 - x_3 + x_4 = 10$$

$$-x_1 + 4x_2 + x_3 - x_5 + x_7 = 6$$

$$x_2 + x_3 + x_6 = 4$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \ge 0$$

The tables obtained by simplex method are as follows

		c_j	–1	2	–1	0	0	0	–M
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
0	y_4	10	3	1	–1	1	0	0	0
–M	y_7	6	–1	[4]	1	0	–1	0	1
0	y_6	4	0	1	1	0	0	1	0
			M + 1	–4M–2	–M+1	0	M	0	0
0	y_4	17/2	13/4	0	–5/4	1	1/4	0	–1/4
2	y_2	3/2	–1/4	1	1/4	0	–1/4	0	1/4
0	y_6	5/2	1/4	0	3/4	0	[1/4]	1	–1/4
			1/2	0	3/2	0	–1/2	0	M+1/2
0	y_4	6	3	0	–2	1	0	–1	0
2	y_2	4	0	1	1	0	0	1	0
0	y_5	10	1	0	3	0	1	4	–1
			1	0	3	0	0	2	M

In the final table the basis is $B = [a_4\ a_2\ a_5]$ and in the initial table the basis is $I = [a_4\ a_7\ a_6]$

$\therefore$ The inverse of the basis in the final table is given by

$$B^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

When b_1 is changed to $b_1 + \Delta b_1$ then the range of Δb_1 such that the optimality of the new BFS is not violated is given by

$$\max\left\{-\frac{x_{Bi}}{b_{i1}} : b_{i1} > 0\right\} \le \Delta b_1 \le \min\left\{-\frac{x_{Bi}}{b_{i1}} : b_{i1} < 0\right\}$$

$$\therefore \max\left\{-\frac{x_{B1}}{b_{11}}\right\} \le \Delta b_1 \quad [\because \text{only } b_{11} = 1 > 0 \text{ and there are no negative } b_{i1}]$$

$$\text{or, } -\frac{6}{1} \le \Delta b_1 \text{ or, } \Delta b_1 \ge -6$$

$\therefore b_1 + \Delta b_1 \geq b_1 - 6$

or, $b_1^* \geq 10 - 6$

or, $b_1^* \geq 4.$

when b_2 is changed to $b_2 + \Delta b_2$ then the range of Δb_2 such that the optimality of the new BFS is not violated is given by

$$\max\left\{-\frac{x_{Bi}}{b_{i2}} : b_{i2} > 0\right\} \leq \Delta b_2 \leq \min\left\{-\frac{x_{Bi}}{b_{i2}} : b_{i2} < 0\right\}$$

$$\therefore \Delta b_2 \leq \min\left\{-\frac{x_{B3}}{b_{i2}}\right\} \left[\because \text{only } b_{32} = -1 < 0 \text{ and there is no positive } b_{i2}\right]$$

$$\text{or, } \Delta b_2 \leq -\frac{10}{-1}$$

or, $\Delta b_2 \leq 10$

$\therefore b_2 + \Delta b_2 \leq b_2 + 10$

$\therefore b_2^* \leq 6 + 10$

or, $b_2^* \leq 16$

When b_3 is changed to $b_3 + \Delta b_3$ then the ranges of Δb_3 such that the optimality of the new BFS is not violated are given by

$$\max\left\{-\frac{x_{Bi}}{b_{i3}} : b_{i3} > 0\right\} \leq \Delta b_3 \leq \min\left\{-\frac{x_{Bi}}{b_{i3}} : b_{i3} < 0\right\}$$

$$\text{or, } \max\left\{-\frac{x_{B2}}{b_{23}}, -\frac{x_{B3}}{b_{33}}\right\} \leq \Delta b_3 \leq \min\left\{-\frac{x_{B1}}{b_{13}}\right\}$$

$$\text{or, } \max\left\{-\frac{4}{1}, \frac{-10}{4}\right\} \leq \Delta b_3 \leq \min\left\{-\frac{6}{-1}\right\}$$

$$\text{or, } -\frac{5}{2} \leq \Delta b_3 \leq 6$$

$$\therefore -\frac{5}{2} + b_3 \leq b_3 + \Delta b_3 \leq 6 + b_3$$

$$\text{or, } -\frac{5}{2}+4 \le b_3^* \le 6+4$$

$$\text{or, } \frac{3}{2} \le b_3^* \le 10$$

Example 103.5.2 : Consider the LPP

$$\text{Maximize} \quad z = 2x_1 + x_2 + 4x_3 - x_4$$

$$\text{subject to} \quad x_1 + 2x_2 + x_3 - 3x_4 \le 8$$

$$-x_2 + x_3 + 2x_4 \le 0$$

$$2x_1 + 7x_2 - 5x_3 - 10x_4 \le 21$$

$$x_1, x_2, x_3, x_4 \ge 0$$

The optimal solution of it is contained in the following table

			2	1	4	–1	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
2	y_1	8	1	0	3	1	1	2	0
1	y_2	0	0	1	–1	–2	0	–1	0
0	y_7	5	0	0	–4	2	–2	3	1
		$z_j - c_j$	0	0	1	1	2	3	0

For each of the parameter change listed below, make the necessary correction in the optimal table and solve the resulting problem.

a) Change c_1 to 1

b) Change c to [1 2 3 4]

c) Change b to $[3 \; -2 \; 4]^T$

d) Change b_2 to 11

e) How much c_1 be changed without affecting the optimal solution.

Solution : a) when c_1 is changed to 1 the midified form of the optimal table becomes

	c_j	1	1	4	−1	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
1	y_1	8	1	0	3	1	1	2	0
1	y_2	0	0	1	−1	−2	0	−1	0
0	y_7	5	0	0	−4	2	−2	3	1
		$z_j - c_j$	0	0	−2	0	1	1	0

From this table we see that the changed solution is not optimal as $z_3 - c_3 < 0$. So we are to apply simplex method to get the optimal solution

	c_j	1	1	4	−1	0	0	0	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	min ratio
1	y_1	8	1	0	[3]	1	1	2	0	8/3
1	y_2	0	0	1	−1	−2	0	−1	0	....
0	y_7	5	0	0	−4	2	−2	3	1	....
			0	0	−2	0	1	1	0	
4	y_3	8/3	1/3	0	1	1/3	1/3	2/3	0	
1	y_2	8/3	1/3	1	0	−5/3	1/3	−1/3	0	
0	y_7	95/3	4/3	0	0	10/3	−2/3	17/3	1	
	$z = 40/3$	$z_j - c_j$	2/3	0	0	2/3	5/3	7/3	0	

since all $z_j - c_j \geq 0$, the optimality conditions are satisfied. The optimal solution is $x_1 = 0, x_2 = 8/3, x_3 = 8/3, z_{max} = 40/3$.

b) When c is changed from [2 1 4 −1] to [1 2 3 4] the modified form of the optimal table becomes

	c_j	1	2	3	4	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
1	y_1	8	1	0	3	1	1	2	0
2	y_2	0	0	1	−1	−2	0	−1	0
0	y_7	5	0	0	−4	2	−2	3	1
		$z_j - c_j$	0	0	−2	−7	1	0	0

we see that there are negative $z_j - c_j$ viz $z_3 - c_3 = -2$ and $z_4 - c_4 = -7$. Hence the solution is not optimal. We apply simplex method to get the optimal solution.

		c_j	1	2	3	4	0	0	0	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	min ratio
1	y_1	8	1	0	3	1	1	2	0	8
2	y_2	0	0	1	–1	–2	0	–1	0	–
0	y_7	5	0	0	–4	[2]	–2	3	1	5/2
		$z_j - c_j$	0	0	–2	–7	1	0	0	
1	y_1	11/2	1	0	[5]	0	2	1/2	–1/2	11/10
2	y_2	5	0	1	–5	0	–2	2	1	–
4	y_4	5/2	0	0	–2	1	–1	3/2	1/2	–
		$z_j - c_j$	0	0	–16	0	–6	21/2	7/2	
3	y_3	11/10	1/5	0	1	0	2/5	1/10	–1/10	
2	y_2	21/2	1	1	0	0	0	5/2	1/2	
4	y_4	47/10	2/5	0	0	1	–1/5	17/10	3/10	
$z = 431/10$		$z_j - c_j$	16/5	0	0	0	2/5	121/10	19/10	

Since all $z_j - c_j \geq 0$ we have obtained the optimal table. The optimal solution is

$x_1 = 0,\ x_2 = 21/12,\ x_3 = 11/10,\ x_4 = 47/10$

and $z_{max} = \frac{431}{10} = 43\frac{1}{10}$.

c) In the initial table the basis is $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [a_5\ a_6\ a_7]$ and so the inverse of the basis of the final table is given by $B^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix}$.

The new solution when b is changed from $[8\ 0\ 21]^T$ to $[3\ -2\ 4]^T$ is given by

$$x_B^* = B^{-1}b^* = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -8 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_7 \end{bmatrix}$$

This solution is not feasible. So the modified form of the final table is not feasible but optimal. Hence to get the optimal solution we are to apply dual simplex method. The following are the modified optimal table and tables obtained by dual simplex method.

		c_j							
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
2	y_1	−1	1	0	3	1	1	2	0
1	y_2	2	0	1	−1	−2	0	−1	0
0	y_7	−8	0	0	[−4]	2	−2	3	1
		$z_j - c_j$	0	0	1	1	2	3	0
$\frac{z_j - c_j}{y_{3j}} : y_{3j} < 0$					−1/4		−1		
2	y_1	−7	1	0	0	5/2	[−1/2]	17/4	3/4
1	y_2	4	0	1	0	−5/2	1/2	−7/4	−1/4
4	y_3	2	0	0	1	−1/2	1/2	−3/4	−1/4
		$z_j - c_j$	0	0	0	3/2	3/2	15/4	1/4
$\frac{z_j - c_j}{y_{2j}} : y_{2j} < 0$							−3		
0	y_5	14	−2	0	0	−5	1	−17/2	−3/2
1	y_2	−3	1	1	0	0	0	5/2	1/2
4	y_3	−5	1	0	1	2	0	7/2	1/2
		$z_j - c_j$	3	0	0	9	0	33/2	5/2

We note here that $x_{B_3} = -5 < 0$ but all $y_{3j} \geq 0$. Hence this changed problem has no feasible solution.

d) When b_2 is changed to 11, the new solution is given by

$$x_B^* = \begin{bmatrix} x_1 \\ x_2 \\ x_7 \end{bmatrix} = B^{-1}b^* = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ -2 & 3 & 1 \end{bmatrix}\begin{bmatrix} 8 \\ 11 \\ 21 \end{bmatrix} = \begin{bmatrix} 30 \\ -11 \\ 38 \end{bmatrix}$$

Since $x_{B_2} = -11 < 0$, the solution is not feasible but optimal. So to get optimal solution we are to apply dual simplex method in the modified optimal table. The dual simplex tables are as follows.

		c_j	2	1	4	−1	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
2	y_1	30	1	0	3	1	1	2	0
1	y_2	−11	0	1	−1	[−2]	0	−1	0
0	y_7	38	0	0	−4	2	−2	3	1
		$z_j - c_j$	0	0	1	1	2	3	0
$\frac{z_j - c_j}{y_{2j}} : y_{2j} < 0$						−1	−1/2		−3	
2	y_1	49/2	1	1/2	5/2	0	1	3/2	0
−1	y_4	11/2	0	−1/2	1/2	1	0	1/2	0
0	y_7	27	0	1	−5	0	−2	2	1
$z = 87/2$		$z_j - c_j$	0	1/2	1/2	0	2	5/2	0

In this table all $x_{B_i} > 0$ and all $z_j - c_j \geq 0$. So we have reached to the optimal table. The optimal solution is $x_1 = 49/2$, $x_2 = 0$, $x_3 = 0$, $x_4 = 11/2$ and $z_{max} = 87/2$

e) When $c_1 = 1$ is replaced by c_1^* the modified form of the optimal table is given by

		c_1^*	1	4	−1	0	0	0
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7
c_1^*	y_1	8	1	0	3	1	1	2	0
1	y_2	0	0	1	−1	−2	0	−1	0
0	y_7	5	0	0	−4	2	−2	3	1
		$z_j - c_j$	0	0	$3c_1^* - 5$	$c_1^* - 1$	c_1^*	$2c_1^* - 1$	0

This table remains as optimal table if all $z_j - c_j \geq 0$

i.e. if $3c_1^* - 5 \geq 0$ and $c_1^* - 1 \geq 0$ and $c_1^* \geq 0$ and $2c_1^* - 1 \geq 0$

i.e. if $c_1^* \geq 5/3$ and $c_1^* \geq 1$ and $c_1^* \geq 0$ and $c_1^* \geq 1/2$

i.e. if $c_1^* \geq 5/3$.

e) Alternative method using formula :

Since $c_1 \in c_B$, the range of Δc_1 for which the optimality of the solution is maintained is given by

$$\max\left\{-\frac{z_j - c_j}{y_{1j}} : y_{1j} > 0\right\} \leq \Delta c_1 \leq \min\left\{-\frac{z_j - c_j}{y_{1j}} : y_{1j} < 0\right\}$$

i.e. $$\max\left\{-\frac{z_3 - c_3}{y_{13}}, -\frac{z_4 - c_4}{y_{14}}, -\frac{z_5 - c_5}{y_{15}}, -\frac{z_5 - c_5}{y_{16}}, \right\} \leq \Delta c_1$$

i.e. $$\max\left\{-\frac{1}{3}, -\frac{1}{1}, -\frac{2}{1}, -\frac{3}{2}\right\} \leq \Delta c_1$$

i.e. $$-\frac{1}{3} \leq \Delta c_1 < \alpha$$

$$\therefore \quad c_1 - \frac{1}{3} \leq c_1 + \Delta c_1 < c_1 + \alpha$$

or, $$2 - \frac{1}{3} \leq c_1^* < \alpha$$

or, $$5/3 \leq c_1^* < \alpha$$

$\therefore$ If $c_1^* \geq 5/3$ the optimal solution remains optimal.

103.6. ADDITION OF A SINGLE VARIABLE

Let the optimal solution of the given LPP

Maximize $z = cx$

subject to $Ax = b,\ x \geq 0$

be known. Let x_{n+1} be added with it and the coefficient vector associated with x_{n+1} be a_{n+1} and the cost coefficient for x_{n+1} be c_{n+1}.

Since b is not changed the old optimal solution will be a feasible solution of the new LPP but it may not be optimal. Let B be the optimal basis and c_B be the associated cost vector of the old LPP. Then they are also the

same for the new LPP. It is optimum for the new LPP if $z_{n+1} - c_{n+1} \geq 0$.

In case $z_{n+1} - c_{n+1} < 0$, x_{n+1} will enter the solution and simplex method is to be applied to the old optimal table added with (n+1)th column as $y_{n+1} = B^{-1}a_{n+1}$.

103.7. ILLUSTRATIVE EXAMPLES

Example 103.7.1 : Consider the LPP

$$\begin{aligned} \text{Maximize} \quad & z = x_1 + 2x_2 + x_3 \\ \text{subject to} \quad & 2x_1 + x_2 - x_3 \leq 2 \\ & 2x_1 - x_2 + 5x_3 \leq 6 \\ & 4x_1 + x_2 + x_3 \leq 6 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Let a new variable $x_3' \geq 0$ be untroduced with cost (i) 3 (ii) 5 and $a_3' = [2 \ -1 \ 4]^T$. Discuss the effect.

The solution of the LPP is obtained by simplex method.

The following are the tables.

		c_j	1	2	1	0	0	0	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	min ratio
0	y_4	2	2	[1]	−1	1	0	0	2
0	y_5	6	2	−1	5	0	1	0	–
0	y_6	6	4	1	1	0	0	1	–
		$z_j - c_j$	−1	−2	−1	0	0	0	
2	y_2	2	2	1	−1	1	0	0	
0	y_5	8	4	0	[4]	1	1	0	2
0	y_6	4	2	0	2	−1	0	1	2
		$z_j - c_j$	3	0	−3	2	0	0	
2	y_2	4	3	1	0	5/4	1/4	0	
1	y_3	2	1	0	1	1/4	1/4	0	
0	y_6	0	0	0	0	−3/2	−1/2	1	
z = 10		$z_j - c_j$	6	0	0	1/4	3/4	0	

$\therefore$ The optimal solution is $x_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$

The inverse of the basis in the optimal table is

$$B^{-1} = \begin{bmatrix} 5/4 & 1/4 & 0 \\ 1/4 & 1/4 & 0 \\ -3/2 & -1/2 & 1 \end{bmatrix}$$

The added column for x_3' is $a_3' = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

The corresponding column in the final table is given by

$$y_3' = B^{-1}a_3' = \begin{bmatrix} 5/4 & 1/4 & 0 \\ 1/4 & 1/4 & 0 \\ -3/2 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 9/4 \\ 1/4 \\ 6/4 \end{bmatrix}$$

(i) when $c_3' = 3$, we have

$$z_3' - c_3' = c_B y_3' - c_3' = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 9/4 \\ 1/4 \\ 6/4 \end{bmatrix} - 3 = \frac{19}{4} - 3 = \frac{7}{4} > 0$$

$\therefore$ The optimality condition is satisfied for the changed problem also. The optimal solution is $x_1 = 0, x_2 = 4, x_3 = 2$.

(ii) when $c_3' = 5$ then we have

$$z_3' - c_3' = c_B y_3' - c_3' = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 9/4 \\ 1/4 \\ 6/4 \end{bmatrix} = \frac{19}{4} - 5 = -\frac{1}{4} < 0$$

$\therefore$ Optimality condition is not satisfied here. We shall modify the optimal table of the old problem with added column $y_3' = \begin{bmatrix} 9/4 \\ 1/4 \\ 6/4 \end{bmatrix}$ and $z_3' - c_3' = -1/4$ and $c_3' = 5$. Then to get the optimal solution we are to apply simplex method. The tables obtained are as follows.

		c_j	1	2	1	5	0	0	0	
c_B	y_B	x_B	y_1	y_2	y_3	y_3'	y_4	y_5	y_6	min ratio
2	y_2	4	3	1	0	9/4	5/4	1/4	0	16/9
1	y_3	2	1	0	1	1/4	1/4	1/4	0	8
0	y_6	0	0	0	0	[6/4]	−3/2	−1/2	1	0→
		$z_j - c_j$	6	0	0	−1/4	11/4	3/4	0	
2	y_2	4	3	1	0	0	7/2	1	−3/2	
1	y_3	2	1	0	1	0	1/2	1/3	−1/6	
5	y_3'	0	0	0	0	1	−1	−1/3	2/3	
$z = 10$		$z_j - c_j$	6	0	0	0	5/2	2/3	1/6	

In this table all $x_{B_i} \geq 0$ and all $z_j - c_j \geq 0$. So the optimality conditions are satisfied. The optimal solution is given by $x_1 = 0$, $x_2 = 4$, $x_3 = 2$, $x_3' = 0$ and $z_{max} = 10$.

103.8. DELETION OF A VARIABLE

From a LPP if we delete a variable then two cases may arise.

Case 1. If the variable deleted is nonbasic then the feasibility and optimality conditions are not affected. So the optimal solution of the old problem is the optimal solution of the new problem.

Case 2. If the variable deleted is basic then the conditions of optimality may be affected and so a new solution is to be obtained. For this new optimal solution, we are to assign a cost-M corresponding to the basic variable to be deleted and apply simplex method after modifying the old optimal table.

103.9. ILLUSTRATIVE EXAMPLES

Example 103.9.1 : For the LPP

$$\text{Maximize} \quad z = x_1 + 2x_2 + x_3$$

$$\text{subject to} \quad 2x_1 + x_2 - x_3 \leq 2$$

$$2x_1 - x_2 + 5x_3 \leq 6$$

$$4x_1 + x_2 + x_3 \leq 6$$

$$x_1, x_2, x_3 \geq 0$$

The optimal table is

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
2	y_2	4	3	1	0	5/4	1/4	0
1	y_3	2	1	0	1	1/4	1/4	0
0	y_6	0	0	0	0	–3/2	–1/2	1
$z = 10$		$z_j - c_j$	6	0	0	1/4	3/4	0

Discuss the effect of deletion of the variable i) x_1 ii) x_2 iii) x_3.

i) From the optimal table we see that x_1 is a non basic variable. So if x_1 is deleted the optimal solution remains unaffected. Hence old optimal solution is also the new optimal solution.

Hence the optimal solution is

$$x_1 = 0,\ x_2 = 4,\ x_3 = 2 \text{ and } z_{max} = 10.$$

ii) From the optimal table we see that x_2 is a basic variable. Hence we make a new starting table by changing $c_2 = 2$ by –M, where M is a big positive number. As M is very large the optimality conditions are not affected and once it goes out from the basis it never reappears in the basis in the simplex method. The modified table is

	c_j		1	–M	1	0	0	0	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	min ratio
–M	y_2	4	[3]	1	0	5/4	1/4	0	4/3
1	y_3	2	1	0	1	1/4	1/4	0	2
0	y_6	0	0	0	0	–3/2	–1/2	1	...
			–3M	0	0	$-\frac{5M}{4}+\frac{1}{4}$	$-\frac{M}{4}+\frac{1}{4}$	0	
1	y_1	4/3	1	1/3	0	5/12	1/12	0	
1	y_3	2/3	0	–1/3	1	–1/6	1/6	0	
0	y_6	0	0	0	0	–3/2	–1/2	1	
$z = 2$		$z_j - c_j$	0	M	0	1/4	1/4	0	

In this table all $z_j - c_j \geq 0$ and all $x_{B_i} \geq 0$, so we have reached to the optimal table. The optimal solution is $x_1 = 4/3,\ x_2 = 0,\ x_3 = 2/3$ and $z_{max} = 2$.

iii) From the optimal table we see that x_3 is a basic variable. Hence we make a new starting table by changing $c_j = 1$ by $-M$, where M is a big positive number. As M is very large the optimality conditions are satisfied. Also once it goes out from the basis it never reappears in the basis in the simplex method. The modified table and other simplex tables are as follows.

		c_j	1	2	$-M$	0	0	0	
c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	min ratio
2	y_2	4	3	1	0	5/4	1/4	0	4/3
$-M$	y_3	2	1	0	1	1/4	1/4	0	2
0	y_6	0	0	0	0	$-3/2$	$-1/2$	1	
			$-M+5$	0	0	$-\frac{M}{4}+\frac{5}{2}$	$-\frac{M}{4}+\frac{1}{2}$	0	
1	y_1	4/3	1	1/3	0	5/12	1/12	0	16
$-M$	y_3	2/3	0	$-1/3$	1	$-1/6$	1/6	0	4
0	y_6	0	0	0	0	$-3/2$	$-1/2$	1	...
1	y_1	1	1	1/2		1/2	0	0	2
0	y_5	4	0	-2		-1	1	0	..
0	y_6	2	0	-1		-2	0	1	...
2	y_2	2	2	1		1	0	0	
0	y_5	8	2	0		$-3/2$	1	0	
0	y_6	4	0	0		$-3/2$	0	1	
		4	3	0		2	0	0	

The optimal table is obtaind and the optimal solution is $x_1 = 0, x_2 = 2, x_3 = 0$ and $z_{max} = 4$.

103.10. ADDITION OF A NEW CONSTRAINT

Addition of a new constraint may or may not affect the current optimal solution. Two cases will arise

i) If the added constraint is satisfied by the old optimal solution then the old optimal solution is also the new optimal solution.

ii) If the added constraint is not satisfied by the old optimal solution, then this old optimal solution becomes

an infeasible solution for the new problem. To obtain the optimal solution for the changed problem we are first to modify the final table and then apply dual simplex method.

The following three situations will arise depending on the nature of the solution to the original LPP.

If the original LPP has an optimal solution then the modified LPP may have an optimal solution or it will give no F.S.

If the original LPP has unbounded solution then the modified LPP may have an optimal solution or it will have no F.S. or it will have unbounded solution.

If the original LPP has no F.S. then the modified LPP will have also no F.S.

103.11. ILLUSTRATIVE EXAMPLES

Example 103.11.1 : Let us consider the final table of a LPP

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
2	y_1	3	1	0	0	–1	0	·5	·2	–1
4	y_2	1	0	1	0	2	1	–1	0	·5
1	y_3	7	0	0	1	–1	–2	5	–·3	2
		$z_j - c_j$	0	0	0	2	0	2	·1	2

where y_6, y_7 and y_8 are slack variables.

If the constraint

i) $2x_1 + 3x_2 - x_3 + 2x_4 + 4x_5 \leq 5$

ii) $2x_1 + 3x_2 - x_3 + 2x_4 + 4x_5 \leq 1$

is added then find the solution of the changed LPP.

Solution :

From the final table we see that the optimal solution of the old LPP is

$x_1 = 3, x_2 = 1, x_3 = 7, x_4 = 0, x_5 = 0, x_6 = 0, x_7 = 0, x_8 = 0$

i) The added constraint is

$$2x_1 + 3x_2 - x_3 + 2x_4 + 4x_5 \leq 5$$

Putting $x_1 = 3, x_2 = 1, x_3 = 7, x_4 = 0, x_5 = 0, x_6 = 0, x_7 = 0, x_8 = 0$ in this constraint we have

$$2.3 + 3.1 - 7 + 2.0 + 4.0 \leq 5$$

or, $6 + 3 - 7 \leq 5$

or, $2 \leq 5$

This is true. So the solution satisfies the added constraint. Hence this old optimal solution is also optimal solution to the new problem.

ii) The added constraint is

$$2x_1 + 3x_2 - x_3 + 2x_4 + 4x_5 \leq 1$$

Putting $x_1 = 3, x_2 = 1, x_3 = 7, x_4 = 0, x_5 = 0, x_6 = 0, x_7 = 0, x_8 = 0$ in this constraint we get

$$2.3 + 3.1 - 7 + 2.0 + 4.0 \leq 1$$

or, $6 + 3 - 2 \leq 1$

or, $2 \leq 1$

This is not true. i.e. the optimal solution to the old problem does not satisfy the added constraint. To get the solution of the new LPP we introduce the new constraint with a new slack variable in the optimal table of the old problem. We then modify this table to have a unit basis and then apply dual simplex method to it. The following are the tables.

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9
2	y_1	3	1	0	0	−1	0	.5	.2	−1	0
4	y_2	1	0	1	0	2	1	−1	0	.5	0
1	y_3	7	0	0	1	−1	−2	5	−.3	2	0
0	y_9	1	2	3	−1	2	4	0	0	0	1
		$z_j - c_j$	0	0	0	2	0	2	.1	2	0
2	y_1	3	1	0	0	−1	0	.5	.2	−1	0
4	y_2	1	0	1	0	2	1	−1	0	.5	0
1	y_3	7	0	0	1	−1	−2	5	−.3	2	0
0	y_9	−1	0	0	0	−3	[−1]	7	−.7	2.5	1
			0	0	0	2	0	2	.1	2	0
$\frac{z_j - c_j}{y_{4j}} : y_{4j} < 0$						$-\frac{2}{3}$	0		$-\frac{.1}{7}$		
2	y_1	3	1	0	0	−1	0	.5	.2	−1	0
4	y_2	0	0	1	0	−1	0	6	−.7	3	1
1	y_3	9	0	0	1	5	0	−9	1.1	−3	−2
2	y_9	1	0	0	0	3	1	−7	.7	−2.5	−1
			0	0	0	2	0	2	.1	2	0

The second table is obtained by the operation $R_4^1 = R_4 - 2R_1 - 3R_2 + R_3$. The third table is obtained by using dual simplex method to the second table and is the final table. The optimal solution is $x_1 = 3, x_2 = 0, x_3 = 9, x_4 = 0, x_5 = 1$.

108.12. SUMMARY

In the introduction the usefulness of post optimality analysis is discussed. Then one by one the different situations viz discrete changes in the cost vector and requirement vector, addition and deletion of a single variable, and addition of a new constraint are discussed. Each situation is illustrated by examples.

108.13. SELF ASSESMENT QUESTIONS

1. For the LPP

$$\begin{aligned} \text{Maximize} \quad & z = 15x_1 + 45x_2 \\ \text{subject to} \quad & 5x_1 + 2x_2 \le 162 \\ & x_1 + 16x_2 \le 240 \\ & x_2 \le 50 \\ & x_1, x_2 \ge 0 \end{aligned}$$

find the optimal solution. Find the range of each cost coefficient (changed one at a time) to give same optimal solution.

$$\left[\text{Ans}: \begin{array}{l} x_1 = 352/13,\ x_2 = 173/13,\ z_{max} = 1005 \\ -195/16 \le \Delta c_1 \le 195/2 \end{array}\right].$$

2. Find how much the 7 in the first constraint of the problem

$$\begin{aligned} \text{Minimize} \quad & z = x_1 - 3x_2 + 2x_3 \\ \text{subject to} \quad & 3x_1 - x_2 + 2x_3 \le 7 \\ & -2x_1 + 4x_2 \le 12 \\ & -4x_1 + 3x_2 + 8x_3 \le 10 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

be changed before the basis of the optimal table would change.

3. Find the optimal solution of the LPP and the separate ranges of variations of b_2 and b_3 consistent with the optimality of the solution

$$\begin{aligned} \text{Maximize} \quad & z = -x_1 + 2x_2 - x_3 \\ \text{subject to} \quad & 3x_1 + x_2 - x_3 \le 10 \end{aligned}$$

$$-x_1 + 4x_2 + x_3 \geq 6$$

$$x_2 + x_3 \leq 4$$

$$x_1, x_2, x_3 \geq 0.$$

Determine also the effect of discrete changes in the components of the cost vector which correspond to the basic variables.

[Ans: $x_1 = 0, x_2 = 4, x_3 = 0; \Delta b_2 \leq 10, -5/2 \leq \Delta b_3 \leq 6, -2 \leq \Delta c_2$]

4. Following is the optimal table for an LPP

		c_j	2	1	1	2	0
c_B	B	x_B	y_1	y_2	y_3	y_4	y_5
2	a_1	3	1	0	–1	3	2
1	a_2	4	0	1	4	–1	–2
			0	0	1	3	2

i) Find the limitations of the values of c_3, c_4, c_5 (taking one at a time) for which the current solution will remain optimal.

ii) Find the optimal solution to the problem, if c_3 is changed to 3.

iii) Find the limitations of the values of c_1 for which the current solution remains optimal

iv) Find the optimal solution to the problem, if c_1 is changed to 5

[Ans: i) $-\infty < c_3 \leq 2, -\infty < c_4 \leq 5, -\infty < c_5 \leq 2$

ii) $x_1 = 4, x_3 = 1, x_2 = 0, x_4 = 0$

iii) $1 \leq c_1 \leq 3$

iv) $x_1 = 13/4, x_2 = 0, x_3 = 1, x_4 = 0$]

5. Find the optimal solution of the LPP

$$\text{Maximize} \quad z = 4x_1 + 3x_2$$

$$\text{subject to} \quad x_1 + x_2 \leq 5$$

$$3x_1 + x_2 \leq 7$$

$$x_1 + 2x_2 \leq 10$$

$$x_1, x_2 \geq 0$$

Show how to find the optimal solution of the problem, if

i) the first component of the original requirement vector be increased by one unit and the third component be decreased by one unit

ii) the second component of the original requirement vector be decreased by two units.

[Ans : i) $x_1 = 1, x_2 = 4, z_{max} = 16$

ii) $x_1 = 0, x_2 = 5, z_{max} = 15$]

6. The optimal simplex table for a maximization problem is

c_B	B	b	y_1	y_2	y_3	y_4	y_5
12	a_2	8/5	0	1	1/5	2/5	−1/5
5	a_1	9/5	1	0	7/5	1/5	2/5
			0	0	17/5	29/5	M − 2/5

where x_3 is surplus, x_4 is slack and x_5 is artificial variable. Let a new variable $x_6 \geq 0$ be introduced in the problem with a cost 18 assigned to it in the objective function. Suppose that the new vector corresponding to the variable x_6 be [3, 2]. Discuss the effect of this addition of a variable on the optimality of the optimal solution of the given problem.

[Ans. New optimal solution is $x_1 = 0$, $x_2 = 4/7$, $z_{max} = 48/7$]

103.14 Sugested further readings

1. Operations Research : Kanti Swarup, P. K. Gupta, Man Mohan Sultan Chand & Sons
2. Linear Programming and Game Theory : Chakraborty & Ghosh; Moulik Library
3. Operations Research : J. K. Sharma; Macmillan India Limited
4. Linear Programming and Theory of Games : S. D. Sharma & Hemlata Sharma; KederNath Ram Nath & Co.
5. Operations Research : R. K. Gupta; Krishna Prakashan Mandir
6. Linear Programming and Theory of Games : P. K. Gupta; Khanna Publishers.

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-II

Paper-IX Special Paper : OR

Module No. - 107

ADVANCED OPTIMIZATION AND OPERATIONS RESEARCH-I
(INTEGER PROGRAMMING)

Module Structure

107.1 Introduction

107.2 Objectives

107.3 Key Words

107.4 IPP at a Glance

107.4.1 Examples of IPP

107.5 Gomory Fractional (all Integers) Algorithm

107.6 Branch-and-Bound Method

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107.9 Self Assessment Questions

107.10 References

2. Objectives : The main objective of this module is to make the reader acquainted with –

- General Quadratic Programming Problem
- Kuhn-Tucker Conditions for QPP
- Wolfe's Modified Simplex Method, and
- Beale's Method.

3. Key words : Quadratic programming problem, Kuhn-Tucker Conditions, complementary slackness condition, Wolfe's modified simplex method, Beale's method.

4. General Quadratic Programming Problem :

Let X^T and $C \in R^n$, and Q be a symmetric $n \times n$ real matrix. Then the problem of maximizing

$$f(X) = CX + \frac{1}{2} X^T Q X$$

subject to $AX \le b$,

$X \ge 0$,

where, $b^T \in R^m$ and A is an $m \times n$ real matrix, is called a general quadratic programming problem.

The function $X^T Q X$ defines a quadratic form with Q being a symmetric matrix. The quadratic form $X^T Q X$ is said to be positive definite (semi-definite) if the values of the principal minor determinants of Q are positive (non-negative). In this case Q is also called positive definite (semi-definite).

Again, $X^T Q X$. is said to be negative definite (semi-definite) if the value of the k-th principal minor determinant of Q has the sign for $(-1)^k$ for $k = 1, 2, n$ (or zero).

Theorem 4.1 : A positive semi-definite (negative semi definite) quadratic form $f(X) = X^T Q X$ is convex (concave) function over R^n, where, $X^T \in R^n$ and Q is an $n \times n$ real matrix.

Proof. Let $X_1^T, X_2^T \in R^n$.

If $X_0 = \lambda X_2 + (1-\lambda) X_1$ for $0 \le \lambda \le 1$, then we have

$$X_0^T Q X_0 = \left[\lambda X_2 + (1-\lambda) X_1\right]^T Q \left[\lambda X_2 + (1-\lambda) X_1\right]$$

$$= \left[X_1 + \lambda (X_2 - X_1)\right]^T Q \left[X_1 + \lambda (X_2 - X_1)\right]$$

$$= X_1^T QX_1 + \lambda(X_2 - X_1)^T QX_1 + \lambda X_1^T Q(X_2 - X_1) + \lambda^2 (X_2 - X_1)^T Q(X_2 - X_1)$$

$$= X_1^T QX_1 + 2\lambda(X_2 - X_1)^T QX_1 + \lambda^2 (X_2 - X_1)^T Q(X_2 - X_1) \quad (4.1)$$

Now since $f(X)$ is positive semi-definite, $X^T QX \geq 0 \forall X$ and so, $\lambda X^T QX \geq \lambda^2 X^T QX$, where $0 \leq \lambda \leq 1$.

Using this in (4.1), we get

$$X_0^T QX_0 \leq X_1^T QX_1 + 2\lambda(X_2 - X_1)^T QX_1 + \lambda(X_2 - X_1)^T Q(X_2 - X_1)$$

$$= X_1^T QX_1 + \lambda(X_2 - X_1)^T QX_1 + \lambda(X_2 - X_1)^T QX_2$$

$$= X_1^T QX_1 + \lambda X_2^T QX_1 - \lambda X_1^T QX_1 + \lambda X_2^T QX_2 - \lambda X_1^T QX_2$$

$$= \lambda X_2^T QX_2 + (1-\lambda) X_1^T QX_1 \quad (\because X_2^T QX_1 = X_1^T QX_2)$$

$$\therefore f(X_0) = f[\lambda X_2 + (1-\lambda) X_1] \leq \lambda f(X_2) + (1-\lambda) f(X_1), 0 \leq \lambda \leq 1.$$

This shows that $f(X)$ is a convex function over R^n.

Theorem 4.2 : A positive definite (negative definite) quadratic form is a strictly convex (strictly concave) function over R^n.

Proof. Same as above.

4.1 Kuhn-Tucker Conditions for QPP

Let us consider a Quadratic Programming Problem in the form

Maximize $z = f(x_1, x_2, ..., x_n) = \sum_{j=1}^{n} c_j x_j + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} c_{jk} x_j x_k$

such that $\sum_{j=1}^{n} a_{ij} x_j \leq b_i, i = 1, 2, ..., m$

$x_j \geq 0, j = 1, 2, ..., n$

where, $c_{jk} = c_{kj} \forall j = 1, 2, ...n;\ k = 1, 2, ..., n;$

and where $b_i \geq 0$ for $i = 1, 2, ..., m$.

Introducing slack variables S_i^2 for $i = 1, 2, ..., m$ and Sm_{m+j}^2 for $j = 1, 2, ..., n$ respectively, the problem becomes

$$\text{Max. } z = f(x) = \sum_{j=1}^{n} c_j x_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} c_{jk} x_j x_k \tag{4.1.1}$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij} x_j + S_i^2 = b_i,\ i = 1, 2, ..., m \tag{4.1.2}$$

$$\text{and } -x_j + S_{m+j}^2 = 0,\ j = 1, 2, ..., n \tag{4.1.3}$$

Now we construct the Lagrangian function as

$$L(X, S; \lambda) = f(X) - \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j - b_i + S_i^2 \right) - \sum_{j=1}^{n} \lambda_{m+j} \left(-x_j + S_{m+j}^2 \right)$$

Thus, Kuhn-Tucker conditions for the problem are given by

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i a_{ij} + \lambda_{m+j} = 0,\ j = 1, 2, ...n. \tag{4.1.4}$$

$$\lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j - b_i \right) = 0,\ i = 1, 2, ..., m. \tag{4.1.5}$$

$$\lambda_{m+j} x_j = 0,\ j = 1, 2, ..., n. \tag{4.1.6}$$

$$\sum_{j=1}^{n} a_{ij} x_j - b_i \le 0,\ i = 1, 2, ..., m. \tag{4.1.7}$$

$$\text{and } \lambda_i, \lambda_{m+j}, x_j \ge 0, i = 1, 2, ..., m, j = 1, 2, ..., n \tag{4.1.8}$$

For a QPP, (4.1.1.) becomes

$$c_j + \sum_{k=1}^{n} c_{jk} x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + \lambda_{m+j} = 0, j = 1, 2, ..., n \tag{4.1.9}$$

Using (4.1.2), (4.1.5) becomes

$$\lambda_i S_i^2 = 0, i = 1, 2, ..., m \tag{4.1.10}$$

Again, combining (4.1.10) & (4.1.6), we have

$$\sum_{i=1}^{m} \lambda_i S_i^2 + \sum_{j=1}^{n} \lambda_{m+j} x_j = 0. \qquad (4.1.11)$$

This condition is called complementary slackness condition.

One important thing to be noted here is that except for the complementary slackness condition, the remaining equations are linear functions in X, λ and S^2. The problem thus becomes equivalent to finding the solution to a set of linear equations which also satisfies the additional condition

$$\sum_{i=1}^{m} \lambda_i S_i^2 + \sum_{j=1}^{n} \lambda_{m+j} x_j = 0.$$

Since $f(X)$ is strictly concave and the solution space is convex, the feasible solution satisfying all these conditions must give the optional solution directly.

Wolfe (1959) suggested a solution procedure for this problem using the ordinary simplex method with slight modification as given in the following section.

5. Wolfe's Modified Simplex Method :

Let the quadratic programming problem be

Maximize $z = f(x) = \sum_{j=1}^{n} c_j x_j + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} c_{jk} x_j x_k$

subject to the constraints $\sum_{j=1}^{n} a_{ij} x_j \le b_i$ and $i = 1, 2, \ldots m; j = 1, 2, \ldots, n.$

where $c_{jk} = c_{kj}$ for all j and k, $b_i \ge 0$ for all $i = 1, 2, \ldots, m$.

Also, assume that the quadratic form $\sum_{j=1}^{n} \sum_{k=1}^{n} c_{jk} x_j x_k$ be negative semi-definite and then $f(X)$ is concave.

Therefore the conditions given in section 4.1 become necessary and sufficient for an optimal solution to the above quadratic programming problem.

Then Wolfe's iterative procedure may be outlined as below:

Step 1 : First, convert the inequality constraints into equations by introducing slack variables S_i^2 for $i = 1, 2, ...,$ and $S_{m+j}^2, j = 1, 2, ..., n$ respectively.

Step 2 : Then, construct the Lagrangian function

$$L(X, S, \lambda) = f(X) - \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j - b_i + S_i^2 \right) - \sum_{j=1}^{n} \lambda_{m+j} \left(-x_j + S_{m+j}^2 \right)$$

Differentiate $L(X, S, \lambda)$ partially with respect to the components of X, S and λ. Equating the first orde partial derivatives to zero, derive Kuhn-Tucker conditions from the resulting equations.

Step 3 : Introduce the non-negative artificial variable w_j,

$j = 1, 2, ..., n$ in the Kuhn-Tucker conditions.

$$c_j + \sum_{k=1}^{n} c_{jk} x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + \lambda_{m+j} + w_j = 0, j = 1, 2, ..., n;$$

and construct the objective function

$$z_w = w_1 + w_2 + ... + w_n.$$

Step 4 : Obtain an initial B.L.S. to the L.P.P.

Min $z_w = w_1 + w_2 + ... + w_n$

or, Max $z_w^* = -w_1 - w_2 - ... - w_n$

subject to $\sum_{k=1}^{n} c_{jk} x_k - \sum_{i=1}^{m} \lambda_j a_{ij} + \lambda_{m+j} + w_j = -c_j, j = 1, 2, ..., n$

$$\sum_{i=1}^{n} a_{ij} x_j + x_{n+i} = b_i, \; i = 1, 2, ..., m$$

$$w_j, \lambda_i, \lambda_{m+j}, x_j \geq 0, \; i = 1, 2, ..., m; j = 1, 2, ... n$$

and satisfying the complementary slackness condition

$$\sum_{j=1}^{n} \lambda_{m+j} x_j + \sum_{i=1}^{m} \lambda_i x_{n+i} = 0,$$

where, $x_{n+i} = S_i^2$, $i = 1, 2, ..., m$

Step 5 : Use two-phase simplex method to obtain an optimal solution of the LPP of Step 4.

This solution is an optimal solution to the given QPP also.

Important remarks on Wolfe's Method :

1. Let us designate as complementary variable each pair of x_j and λ_{m+j} and like wise each pair of S_i^2 and λ_i.

   In this method the basic solution at any iteration may correspond to one of the two cases:

   I. For each j and each i the basis contains only one complementary variable. Such a basic solution is called a standard basic solution and satisfies the complementary slackness condition.

   II. For each j and each i the basis contains a basic pair of complementary variables. Such a basic solution is called a non-standard basic solution and may not satisfy the complementary slackness condition. Whenever a non-standard basic occurs, the selection procedure for entering variable into the basis seeks to re-establish the complementary slackness condition.

2. If the quadratic programming problem is given in the minimization form, then convert it into maximization one by suitable modification in $f(X)$ and the '$\geq$' constraints.

3. The solution of the above system is obtained by using Phase-I of two phase simplex method. Since our aim is to obtain a feasible solution, the solution does not require the consideration of Phase-II. The only necessary thing is to maintain the conditions $\lambda_i S_i^2 = 0 = \lambda_{m+j} x_j$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$ all the time. This implies that if λ_i is in the basic solution with positive value, then S_i^2 can not present in the basis with positive value. Similarly, λ_{m+j} and x_j can not be positive simultaneously.

4. It should be noted that Phase-I will end in the usual manner with the sum of all artificial variables equals to zero only if the feasible solution exists.

5.1 Illustrative examples :

Example 5.1.1. Apply Wolfe's method for solving the QPP:

$$\text{Max.} Z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$$

subject to $x_1 + 2x_2 \le 2$ and

$x_1, x_2 \ge 0.$

Solution.

Step 1. First, we write all the constraint inequalities with '$\le$' sign as follows :

$x_1 + 2x_2 \le 2, -x_1 \le 0, -x_2 \le 0.$

Introducing slack variables S_1^2, S_2^2, S_3^2, the given problem becomes of the form :

$$\text{Max.} Z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$$

subject to $x_1 + 2x_2 + S_1^2 = 2$

$-x_1 + S_2^2 = 0$

$-x_2 + S_3^2 = 0.$

Step 2. We construct the Lagrangian function as

$$L(X, S, \lambda) = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$$
$$-\lambda_1\left(x_1 + 2x_2 - 2 + S_1^2\right) - \lambda_2\left(-x_1 + S_2^2\right) - \lambda_3\left(-x_2 + S_3^2\right).$$

Kuhn-Tucker conditions, are thus given by

$$\frac{\partial L}{\partial x_1} = 0, \frac{\partial L}{\partial x_2} = 0;$$

$$\frac{\partial L}{\partial S_1} = 0, \frac{\partial L}{\partial S_2} = 0, \frac{\partial L}{\partial S_3} = 0;$$

and $\frac{\partial L}{\partial \lambda_1} = 0, \frac{\partial L}{\partial \lambda_2} = 0, \frac{\partial L}{\partial \lambda_3} = 0.$

which give

$$4 - 4x_1 - 2x_2 - \lambda_1 + \lambda_2 = 0,$$

$$6 - 2x_1 - 4x_2 - 2\lambda_1 + \lambda_3 = 0,$$

$$-2\lambda_1 S_1 = 0$$

$$-2\lambda_2 S_2 = 0$$

$$-2\lambda_3 S_3 = 0$$

$$-\left(x_1 + 2x_2 - 2 + S_1^2\right) = 0$$

$$-\left(x_1 + S_2^2\right) = 0$$

$$-\left(-x_2 + S_3^2\right) = 0$$

Kuhn-Tucker conditions also give

$$\lambda_1\left(x_1 + 2x_2 - 2\right) = 0, \text{ i.e., } -\lambda_1 S_1^2 = 0, \text{ or, } \lambda_1 S_1^2 = 0,$$

$$-x_1\lambda_2 = 0; \text{ or, } \lambda_2 x_1 = 0,$$

$$-x_2\lambda_3 = 0, \text{ or, } \lambda_3 x_2 = 0,$$

$$S_1^2, \lambda_1, \lambda_2, \lambda_3, x_1 x_2 \geq 0.$$

Thus,

$$4x_1 + 2x_2 + \lambda_1 - \lambda_2 = 4$$

$$2x_1 + 4x_2 + 2\lambda_1 - \lambda_3 = 6$$

$$x_1 + 2x_2 + S_1^2 = 2$$

and the complementary slackness condition is

$$\lambda_1 S_1^2 + \lambda_2 x_1 + \lambda_3 x_2 = 0,$$

$$S_1^2, \lambda_1, \lambda_2, \lambda_3, x_1, x_2 \geq 0.$$

Step 3 : Introducing artificial variables w_1 and w_2, we get

$$\text{Max.} Z_w^* = -w_1 - w_2$$

$$\text{subject to} \quad 4x_1 + 2x_2 + \lambda_1 - \lambda_2 + w_1 = 4$$

$$2x_1 + 4x_2 + 2\lambda_1 - \lambda_3 + w_2 = 6$$

$$x_1 + 2x_2 + S_1^2 = 2$$

where all variables are non-negative and $\lambda_1 S_1^2 = 0, \lambda_2 x_1 = 0, \lambda_3 x_2 = 0$ altogether give the complementary slackness condition.

Now, all these contraint-equations can be written in matrix form as follows:

$$\begin{bmatrix} 4 & 2 & 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 4 & 2 & 0 & -1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ w_1 \\ w_2 \\ S_1^2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}$$

Step 4 : An initial B.F.S. of the above L.P.P. is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ S_1^2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}$$

Step 5 : The initial table for Phase-I is obtained as given in the following Table 5.1.1.

		$c_j \rightarrow$	0	0	0	0	0	–1	–1	0
C_B	B	x_B	x_1	x_2	λ_1	λ_2	λ_3	w_1	w_2	S_1^2
–1	w_1	4	4	2	1	–1	0	1	0	0
–1	w_2	6	2	4	2	0	–1	0	1	0
0	S_1^2	2	1	2	0	0	0	0	0	1
$\Delta_j = z_j - c_j \rightarrow$			–6	–6	–3	1	1	0	0	0

Table 5.1.1

Here, we compute $\Delta_1 = (-1,-1,0)(4,2,1) - 0 = 6,$

$\Delta_2 = (-1,-1,0)(2,4,2) - 0 = -6,$ etc.

Now we enter to first iteration as follows.

First iteration : Since $\lambda_2 = 0, x_1$ is introduced into the basic solution with w_1 as the leaving variable. It should be noticed that λ_1 can not enter into the basis, because S_1^2 is a basic variable. This gives the following transformed table by our usual rules of transformation.

		$c_j \rightarrow$	0	0	0	0	0	−1	−1	0
C_B	B	x_B	x_1	x_2	λ_1	λ_2	λ_3	w_1	w_2	S_1^2
0	x_1	1	1	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	0	0
−1	w_2	4	0	3	$\frac{3}{2}$	$\frac{1}{2}$	−1	$-\frac{1}{2}$	1	0
0	S_1^2	1	0	$\boxed{\frac{3}{2}}$	$-\frac{1}{4}$	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	1
$\Delta_j = z_j - c_j \rightarrow$			0	−3 ↑	$-\frac{3}{2}$	$-\frac{1}{2}$	1	$\frac{3}{2}$	0	0 ↓

Table 5.1.2

Second iteration : Since $\lambda_3 = 0, x_2$ is introduced into the basic solution with S_1^2 as the leaving variable. By usual rules of matrix transformation, we get next improved table.

		$c_j \rightarrow$	0	0	0	0	0	−1	−1	0
C_B	B	x_B	x_1	x_2	λ_1	λ_2	λ_3	w_1	w_2	S_1^2
0	x_1	$\frac{2}{3}$	1	0	$\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	0	$-\frac{1}{3}$
−1	w_2	2	0	0	$\boxed{2}$	0	−1	0	1	−2
0	x_2	$\frac{2}{3}$	0	1	$-\frac{1}{6}$	$\frac{1}{6}$	0	$-\frac{1}{6}$	0	$\frac{2}{3}$
$\Delta_j = z_j - c_j \rightarrow$			0	0	−2 ↑	0	1	1	0 ↓	2

Table 5.1.3

Third iteration : Since $S_1^2 = 0$, hence λ_1 can be introduced into the basic solution. The improved table of third iteration is given below.

		$c_j \rightarrow$	0	0	0	0	0	−1	−1	0
C_B	B	x_B	x_1	x_2	λ_1	λ_2	λ_3	w_1	w_2	S_1^2
0	x_1	$\frac{1}{3}$	1	0	0	$-\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	0
−1	λ_1	1	0	0	1	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	−1
0	x_2	$\frac{5}{6}$	0	1	0	$\frac{1}{6}$	$-\frac{1}{12}$	$-\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{2}$
$\Delta_j = z_j - c_j \rightarrow$			0	0	0	0	0	1	1	0

Table 5.1.4

Here all Δ_j are non-negative. Hence Table 5.1.4 gives us the optimal solution for Phase-I.

Again, since neither of the artificial variables are present in the basis, the solution is feasible also.

Hence, this solution is optimal solution to the given QPP.

Thus the required optimal solution is given by $x_1^* = \frac{1}{3}, x_2^* = \frac{5}{6}$ and the optimal value z^* is obtained from the original objective function as $z^* = 4\left(\frac{1}{3}\right) + 6\left(\frac{5}{6}\right) - 2\left(\frac{1}{3}\right)^2 - 2\left(\frac{1}{3}\right)\left(\frac{5}{6}\right) - 2\left(\frac{5}{6}\right)^2 = \frac{25}{6}$.

Example 5.1.2. Apply Wolfe's method to solve the quadratic programming problem :

$$\text{Max. } Z = 2x_1 + x_2 - x_1^2$$

$$\text{subject to } 2x_1 + 3x_2 \le 6$$

$$2x_1 + x_2 \le 4$$

$$\text{and } x_1, x_2 \ge 0.$$

Solution.

Step 1 : Writing all the constraint enequalities with '$\leq$' sign we obtain

$$2x_1 + 3x_2 \leq 6, 2x_1 + x_2 \leq 4, -x_1 \leq 0, -x_2 \leq 0.$$

Introducing slcak variables $S_1^2, S_2^2, S_3^2, S_4^2$ the given problem becomes

Max. $Z = 2x_1 + x_2 - x_1^2$

subject to $2x_1 + 3x_2 - S_1^2 = 6$

$$2x_1 + x_2 + S_2^2 = 4$$

$$-x_1 + S_3^2 = 0$$

$$-x_2 + S_4^2 = 0.$$

Step 2 : Now we construct the Lagrangian function as

$$L(X, S, \lambda) = (2x_1, x_2 - x_1^2) - \lambda_1(2x_1 + 3x_2 + S_1^2 - 6) - \lambda_2(2x_1 + x_2 + S_2^2 - 4)$$

$$-\lambda_3(-x_1 + S_3^2) - \lambda_4(-x_2 + S_4^2).$$

Necessary and sufficient conditions are given by

$$\frac{\partial L}{\partial x_1} = 0, \text{ or, } 2 - 2x_1 - 2\lambda_1 - 2\lambda_2 + \lambda_3 = 0;$$

$$\frac{\partial L}{\partial x_2} = 0, \text{ or, } 1 - 3\lambda_1 - \lambda_2 + \lambda_4 = 0;$$

$$\frac{\partial L}{\partial \lambda_1} = 0, \text{ or, } 2x_1 + 3x_2 + S_1^2 - 6 = 0;$$

$$\frac{\partial L}{\partial \lambda_2} = 0, \text{ or, } 2x_1 + x_2 + S_2^2 - 4 = 0$$

along with the complementary slackness condition

$$\lambda_1 S_1^2 + \lambda_2 S_2^2 + \lambda_3 x_1 + \lambda_4 x_2 = 0$$

where, $x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, S_1^2, S_2^2, S_3^2, S_4^2 \geq 0.$

Step-3 : Introducing artificial variables w_1 and w_2, the modified linear programming problem is given by

Max. $z_w^* = -w_1 - w_2$

subject to $2x_1 + 2\lambda_1 + 2\lambda_2 - \lambda_3 + w_1 = 2$

$3\lambda_1 + \lambda_2 - \lambda_4 + w_2 = 1$

$2x_1 + 3x_2 + S_1^2 = 6$

$2x_1 + x_2 + S_2^2 = 4$

$x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, S_1^2, S_2^2, S_3^2, S_4^2 \geq 0.$

satisfying the complementary slackness condition

$\lambda_1 S_1^2 + \lambda_2 S_2^2 + \lambda_3 x_1 + \lambda_4 x_2 = 0.$

The above set of equations can be written in matrix form as below:

$$\begin{bmatrix} 2 & 0 & 2 & 2 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ w_1 \\ w_2 \\ S_1^2 \\ S_2^2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 6 \\ 4 \end{bmatrix}$$

Step - 4 : An initial B.F.S. of the above L.P.P. is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ S_1^2 \\ S_2^2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 6 \\ 4 \end{bmatrix}$$

Step-5 : The initial table for Phase-I is obtained as the following Table 5.1.5.

	$c_j \rightarrow$		0	0	0	0	−1	−1	0	0	0	0
C_B	B	x_B	x_1	x_2	λ_1	λ_2	λ_3	λ_4	w_1	w_2	S_1^2	S_2^2
−1	w_1	2	**2**	0	2	2	−1	0	1	0	0	0
−1	w_2	1	0	0	3	1	0	−1	0	1	0	0
0	S_1^2	6	2	3	0	0	0	0	0	0	1	0
0	S_2^2	4	2	1	0	0	0	0	0	0	0	1
$\Delta_j = z_j - c_j \rightarrow$			−2	0	−5	−3	1	1	0	0	0	0

Table 5.1.5

Now we start the first iteration.

First iteration : Since $\lambda_3 = 0, x_1$ is introduced into the basic solution with w_1 as the leaving variables. λ_1 and λ_2 can not be introduced because S_1^2 and S_2^2 are basic variables.

This gives the transformed Table 5.1.6.

	$c_j \rightarrow$		0	0	0	0	−1	−1	0	0	0	0
C_B	B	x_B	x_1	x_2	λ_1	λ_2	λ_3	λ_4	w_1	w_2	S_1^2	S_2^2
0	x_1	1	1	0	1	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0
−1	w_2	1	0	0	3	1	0	−1	0	1	0	0
0	S_1^2	4	0	**3**	−2	−2	1	0	0	0	1	0
0	S_2^2	2	0	1	−2	−2	1	0	−1	0	0	1
$\Delta_j = z_j - c_j \rightarrow$			0	0	−3	−1	0	0	1	1	0	0

Table 5.1.6

Now we enter to the second iteration.

Second iteration : Again since S_1^2 and S_2^2 are in the basis, neither λ_1 nor λ_2 can replace w_2 in the basis. But, since $\lambda_4 = 0, x_2$ can be introduced in the basis with S_1^2 as the leaving variable. Thus we get the Table 5.1.7 as the transformed table.

		$c_j \rightarrow$	0	0	0	0	0	0	−1	−1	0	0
C_B	B	x_B	x_1	x_2	λ_1	λ_2	λ_3	λ_4	w_1	w_2	S_1^2	S_2^2
0	x_1	1	1	0	1	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0
−1	w_2	1	0	0	3	1	0	−1	0	1	0	0
0	x_2	$\frac{4}{3}$	0	1	$-\frac{2}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	0
0	S_2^2	$\frac{2}{3}$	0	0	$-\frac{4}{3}$	$-\frac{4}{3}$	$\frac{2}{3}$	0	$-\frac{2}{3}$	0	$-\frac{1}{3}$	1
$\Delta_j = z_j - c_j \rightarrow$			0	0	−3	−1	0	1	1	0	0	0

Table 5.1.7

Now we enter to the next iteration.

Third iteration : Now since $S_1^2 = 0, \lambda_1$ can be introduced in the basis with w_2 as the leaving variable. This gives the transformed Table 5.1.8.

		$c_j \rightarrow$	0	0	0	0	0	0	−1	−1	0	0
C_B	B	x_B	x_1	x_2	λ_1	λ_2	λ_3	λ_4	w_1	w_2	S_1^2	S_2^2
0	x_1	$\frac{2}{3}$	1	0	0	$\frac{2}{3}$	$-\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{3}$	0	0
0	λ_1	$\frac{1}{3}$	0	0	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	0	0
0	x_2	$\frac{14}{9}$	0	1	0	$-\frac{4}{9}$	$\frac{1}{3}$	$-\frac{2}{9}$	$-\frac{1}{3}$	$\frac{2}{9}$	$\frac{1}{3}$	0
0	S_2^2	$\frac{10}{9}$	0	0	0	$\frac{8}{9}$	$\frac{2}{3}$	$-\frac{4}{9}$	$-\frac{2}{3}$	$\frac{4}{9}$	$-\frac{1}{3}$	1
$\Delta_j = z_j - c_j \rightarrow$			0	0	0	0	0	0	1	1	0	0

Table 5.1.8

Since all $\Delta_j \geq 0$ and both the artificial variables w_1, w_2 are out of the basis, the computation is now complete. The optimal solution is

$$x_1^* = 2/3, x_2^* = 14/9, \lambda_1 = 1/3, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0,$$

$$w_1 = 0, w_2 = 0, S_1^2 = 0 \text{ and } S_2^2 = 10/9$$

and the maximum value is $Z^* = 2\left(2/3\right) + 14/9 - 2/3 = 22/9$.

6.Beale's Method :

Another approach to solve a quadratic programming problem has been suggested by Beale (1959). This is a technique for solving QPP without using the Kuhn–Tucker conditions. It involves the partitioning the variables into basis and non-basic ones. At each iteration, the objective function is expressed in terms of the non-basic variables only.

Let the QPP be given in the form

Min. $f(X) = CX + \frac{1}{2}X^T QX$, subject to the constraints $AX = b, X \geq 0$. Without loss of generality, every QPP with linear constraints can be written in this form.

The Beale's iterative procedure for solving such type of QPP can be outlined in the following steps.

Step-1. Convert the maximization problem into that of minimization, if it is in maximization form. Introduce slack and/or surplus variables to convert the inequalities into strict equations.

Step-2. Start with any initial solution as the basic solution and let it be denoted by

$$X_B = \left(x_{B_1}, x_{B_2}, ..., x_{B_m}\right).$$

In fact, we can choose arbitrarily any of the m variables as basic ones so that the remaining $n-m$ variables become non-basic. Let the non-basic variables be denoted by

$$X_{NB} = \left(x_{NB_1}, x_{NB_2}, ..., x_{NB_n}\right).$$

Step-3. Express the basic variables $x_{B_1}, x_{B_2}, ..., x_{B_m}$ in terms of the non-basic variables $x_{NB_1}, x_{NB_2}, ..., x_{NB_{n-m}}$ only, using the given and additional constraints, if any.

Step-4. Express $f(X)$ in terms of X_{NB} only, using the given and additional constraints, if any.

Step-5. Evaluate the partial derivatives of $f(X)$ with respect to the non-basic variables at $X_{NB} = 0$.

Step-6. See the nature of $\left(\frac{\partial f}{\partial x_{NB_k}}\right)_{X_{NB}=0}, k = 1, 2, ..., n-m.$

(i) If $\left(\frac{\partial f}{\partial x_{NB_k}}\right)_{X_{NB}=0} \geq 0$ for all $k = 1, 2, ..., n-m$, then the current solution is optimal.

(ii) If $\left(\frac{\partial f}{\partial x_{NB_k}}\right)_{X_{NB}=0} < 0$ for at least one k, then the current solution is not optimal. The non-basic variable corresponding to the maximum of $\left|\left(\frac{\partial f}{\partial x_{NB_k}}\right)_{X_{NB}=0}\right|$ will enter the basis.

Step-7. Let x_i be the entering variable identified in the above step. Determine how much x_i can be increased by calculating

(a) the largest value of x_i that can be attained without deriving the present basic variable (say x_t) negative.

(b) that value of x_i at which $\frac{\partial f}{\partial x_{NB_i}}$ vanishes.

Step-8. Choose the minimum of the values obtained in (a) and (b) above. If it occurs for (a), x_t will leave the basis. If it occurs for (b), no basic variable can be removed from the basis. In this case, we enlarge X_{NB} by introducing a new unrestricted variable u_i called a free variable defined by $u_i = \frac{1}{2}\frac{\partial f}{\partial x_i}$.

This will replace the non-basic variable x_i that has become a new basic variable. The equation $u_i = \frac{1}{2}\frac{\partial f}{\partial x_i}$ is to be treated as as additional constraint.

Step-9. Got to step-3 and repeat the procedure until an optimal solution is reached. If X_{NB} contains a free variable u_i, then the optimality criteria includes an extra condition $\left(\frac{\partial f}{\partial u_i}\right)_{X_{NB}=0, u_i=0} = 0 \forall i.$

Convergence : We give here a feasibility argument for the convergence of Beale's method in a finite number of steps. We shall say that the objective function is in standard form if the linear term contains no free variable.

For convergence we need the following procedure : If, at any stage, there is a non-basic free variable u for which $\frac{\partial f}{\partial y} = 0$, then a free variable must be removed from the non-basic set. Now, when f is in standard form its value in the present trial solution is a necessary value of f subject to the restriction that all the present non-basic restricted variables take the value zero, so there can be only one such value for any set of non-basic restricted variables. There are only a finite number of possible sets of non-basic restricted variables, so in order to show that the iterations, if not initially in standard form, must terminate, it suffices to show that they invariably reach a standard form in a finite number of steps.

We prove this as follows: our procedure ensures that when f is not in standard form a free variable will be removed from the non-basic set. Let p be the number of non-basic free variables. It is easy to show that, in the new expression for f, the off-diagonal elements in the row associated with the new non-basic variable must vanish.

It follows that f does not contain a linear term in this variable, and furtermore f can never contain a linear term in the variables unless some other restricted variable becomes non-basic, thereby decreasing p.

Therefore, if f is not in standard form, and $p = p_0$, then p can not increase and must decrease after at most p_0 steps, unless p meanwhile achieves a standard form. Since f is always in standard form when p=0, the required result follows.

6.1 Illustrative Examples :

Example 6.1.1. Use Beale's method for solving the quadratic programming problem

Maximize $Z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$

subject to $x_1 + 2x_2 \leq 2$

and $x_1, x_2 \geq 0$.

Solution. The given problem is a maximization problem.

So, we convert it to the following minimization problem at first.

Minimize $f(x_1, x_2) = -4x_1 - 6x_2 + 2x_1^2 + 2x_1x_2 + 2x_2^2$

subject to $x_1 + 2x_2 \leq 2$

and $x_1, x_2 \geq 0$.

Next we inttroduce slack variable x_3 to obtain the strict equality constraint

$x_1 + 2x_2 + x_3 = 2,$

where, $x_1, x_2, x_3 \geq 0$.

Now we choose arbitrarily x_1 as the basic variable, so that $X_B = (x_1)$ and $X_{NB} = (x_2, x_3)$.

The only basic variable x_1 can be expressed in terms of the non-basic variables x_2 and x_3 as

$x_1 = 2 - 2x_2 - x_3.$

Also we express f in terms of the non-basic variables as

$$f(x_2, x_3) = -4(2 - 2x_2 - x_3) - 6x_2 + 2(2 - 2x_2 - x_3)^2 + 2(2 - 2x_2 - x_3)x_2 + 2x_2^2$$

$$\therefore \frac{\partial f}{\partial x_2} = 8 - 6 + 4(2 - 2x_2 - x_3)(-2) + 2(2 - 2x_2 - x_3) + 2(-2)x_2 + 4x_2$$

$$= 12x_2 + 6x_3 - 10$$

and $$\frac{\partial f}{\partial x_3} = 4 + 4(2 - 2x_2 - x_3)(-1) - 2x_2$$

$$= 6x_2 + 4x_3 - 4$$

$$\therefore \left(\frac{\partial f}{\partial x_2}\right)_{x_2=0,x_3=0} = 8-6+4(2)(-2)+2(2) = -10$$

and $$\therefore \left(\frac{\partial f}{\partial x_2}\right)_{x_2=0,x_3=0} = 4+4(2)(-1) = -4$$

$\therefore$ the current solution is not optimal.

Clearly, x_2 will enter the basis.

We must now determine how much x_2 should or may increase. The maximum value of x_2 allowed to attain is determined by checking two quantities -

a) the largest value of x_2 that can be attained without deriving the present basic variable negative.

Since $x_1 = 2-2x_2-x_3$ and $x_3 = 0, x_1$ will become negative if x_2 is chosen greater than 1.

b) the value of x_2 for which $\frac{\partial f}{\partial x_2}$ vanishes.

Since $\frac{\partial f}{\partial x_2} = 12x_2 + 6x_3 - 10$, then it vanishes for $x_2 = \frac{5}{6}$ (since $x_3 = 0$).

The minimum of these two values of x_2 is $\frac{5}{6}$ which is obtained in case (b). So, x_1 can not be removed from the basis. In this case, we enlarge X_{NB} by introducing the free variable $u_2 = \frac{1}{2}\frac{\partial f}{\partial x_2} = 6x_2 + 3x_3 - 5$

or, $$x_2 = \frac{1}{6}(u_2 - 3x_3 + 5)$$

Now we start the next iteration with. $X_B = (x_1, x_2)$ and $X_{NB} = (u_2, x_3)$.

Expressing x_1 in terms of u_2 and x_3, we have

$$x_1 = 2 - \frac{1}{3}(u_2 - 3x_3 + 5) - x_3$$

$$= \frac{1}{3} - \frac{1}{3}u_2.$$

Also $x_2 = \frac{1}{6}(u_2 - 3x_3 + 5)$

$$\therefore f(u_2, x_3) = -4\left(\frac{1}{3} - \frac{1}{3}u_2\right) - (u_2 - 3x_3 + 5) + 2\left(\frac{1}{3} - \frac{1}{3}u_2\right)^2$$

$$+2\left(\frac{1}{3} - \frac{1}{3}u_2\right)\cdot\frac{1}{6}(u_2 - 3x_3 + 5) + 2\cdot\frac{1}{36}(u_2 - 3x_3 + 5)^2$$

$$\therefore \frac{\partial f}{\partial u_2} = \frac{4}{3} - 1 + 4\left(\frac{1}{3} - \frac{1}{3}u_2\right) - \left(-\frac{1}{3}\right) + 2\left(-\frac{1}{3}\right)\cdot\frac{1}{6}(u_2 - 3x_3 + 5) + 2\left(\frac{1}{3} - \frac{1}{3}u_2\right)\cdot\frac{1}{6} + \frac{1}{9}(u_2 - 3x_3 + 5)$$

and $\frac{\partial f}{\partial x_3} = 3 + 2\left(\frac{1}{3} - \frac{1}{3}u_2\right)\cdot\frac{1}{6}(-3) + \frac{1}{9}(u_2 - 3x_3 + 5)(-3)$

$$\therefore \left(\frac{\partial f}{\partial u_2}\right)_{u_2=0, x_3=0} = \frac{4}{3} - 1 - \frac{4}{9} - \frac{5}{9} + \frac{1}{9} + \frac{5}{9} = 0$$

and $\left(\frac{\partial f}{\partial x_3}\right)_{u_2=0, x_3=0} = 3 - \frac{1}{3} - \frac{5}{3} = 1 > 0$

Hence the optimality condition is reached and the optimal solution is

$$x_1^* = \frac{1}{3}, x_2^* = \frac{5}{6}, x_3 = 0, u_2 = 0$$

and so, $Z_{max} = -f_{min} = \frac{25}{6}$.

Example 6.1.2. Solve the following quadratic programming problem by Beale's method.

Maximize $Z = 2x_1 + 3x_2 - x_1^2$

subject to $x_1 + 2x_2 \le 4$,

and $x_1, x_2 \ge 0$.

Solution. At first, we convert the given maximization problem into the following minimization problem.

Minimize $f(x_1, x_2) = -2x_1 - 3x_2 + x_1^2$

subject to $x_1 + 2x_2 \le 4, x_1, x_2 \ge 0.$

Introducing the slack variable x_3, the given constraint becomes

$$x_1 + 2x_2 + x_3 = 4,$$

$$x_1, x_2, x_3 \ge 0.$$

Now we arbitrarily choose x_1 as basic variable, so that

$$X_B = (x_1) \text{ and } X_{NB} = (x_2, x_3)$$

The basic variable x_1 can be expressed in terms of the non-basic variables x_2 and x_3 as

$$x_1 = 4 - 2x_2 - x_3$$

$$\therefore f(x_2, x_3) = -2(4 - 2x_2 - x_3) - 3x_2 + (4 - 2x_2 - x_3)^2$$

$$\therefore \frac{\partial f}{\partial x_2} = 4 - 3 + 2(4 - 2x_2 - x_3)(-2)$$

$$= 8x_2 + 4x_3 - 15$$

and $$\frac{\partial f}{\partial x_3} = +2 - 2(4 - 2x_2 - x_3)$$

$$= 4x_2 + 2x_3 - 6$$

$$\therefore \left(\frac{\partial f}{\partial x_2}\right)_{x_2=0, x_3=0} = -15$$

and $$\left(\frac{\partial f}{\partial x_3}\right)_{x_2=0, x_3=0} = -6$$

$\therefore$ the current solution is not optimal. The maximum among the absolute values of the above two partial derivatives is 15 and it is corresponding to the non-basic variable x_2. So, x_2 will enter the basis.

In order to determine how much x_2 should or may be increased, we calculate the following two quantities:

a) the largest value of x_2 that can be attained without deriving the present basic variable negative.

Since $x_1 = 4 - 2x_2 - x_3$ and $x_3 = 0, x_1$ will become negative if x_2 is chosen greater than 2.

b) the value of x_2 for which $\frac{\partial f}{\partial x_2}$ vanishes.

The minimum of these two values of x_2 is $\frac{15}{8}$ which is obtained in case (b). So, x_1 can not be removed from the basis. In this case, we enlarge X_{NB} by introducing the free variable

$$u_2 = \frac{1}{2}\frac{\partial f}{\partial x_2} = -\frac{15}{2} + 4x_2 + 2x_3$$

or, $$x_2 = \frac{1}{4}\left(u_2 - 2x_3 + \frac{15}{2}\right)$$

Now we start the next iteration with $X_B = (x_1, x_2)$ and $X_N = (u_2, x_3)$.

Expressing x_1 in terms of u_2 and x_3, we have

$$x_1 = 4 - \frac{1}{2}\left(u_2 - 2x_3 + \frac{15}{2}\right) - x_3$$

$$= \frac{1}{4} - \frac{1}{2}u_2$$

Also, $x_2 = \frac{1}{4}\left(u_2 - 2x_3 + \frac{15}{2}\right)$

$$\therefore f(u_2, x_3) = -2\left(\frac{1}{4} - \frac{1}{2}u_2\right) - 3\left(\frac{1}{4}u_2 - \frac{1}{2}x_3 + \frac{15}{3}\right) + \left(\frac{1}{4} - \frac{1}{2}u_2\right)^2$$

$$= \frac{1}{4}u_2^2 + \frac{3}{2}x_3 - \frac{97}{16}$$

$$\therefore \frac{\partial f}{\partial u_2} = \frac{1}{2}u_2$$

and $\frac{\partial f}{\partial x_3} = \frac{3}{2}$

$$\therefore \left(\frac{\partial f}{\partial u_2}\right)_{u_2=0, x_3=0} = 0$$

and $\left(\frac{\partial f}{\partial x_3}\right)_{u_2=0, x_3=0} = \frac{3}{2} > 0$.

Hence the optimality condition is reached and the optimal solution is $x_1^* = \frac{1}{4}, x_2^* = \frac{15}{8}, x_3^* = 0, u_2^* = 0$ and

$$z_{max} = -f_{min} = +2\left(\frac{1}{4}\right) + 3\left(\frac{15}{8}\right) - \left(\frac{1}{4}\right)^2 = \frac{97}{16}$$

7. Summary :

General qudratic programming problem is discussed here. The Kuhn-Tucker necessary and sufficient conditions for constrained maxima or minima of a quadratic objective function are also obtained here. Two commonly used method to solve quadratic programming problem, namely, Wolfe's modified simplex method and Beale's method are described here.

Wolfe's method is based on fairly simple modification of simplex method and converges in a finite number of iterations. In this method, the inequality constraints, as well as the non-negativity constraints are converted into equations by introducing slack variables. Then the Lagrangian function is constructed and the Kuhn-Tucker conditions

are obtained. Based on this conditions, an LPP is formed. Two phase simplex method is used to obtain an optimum solution to the LPP formed so, the solution satisfying an additional condition, called the complementary slackness condition. The optimum solution to this LPP is an optimum solution to the given quadratic programming problem also.

Beale's method is a technique of solving the quadratic programming problem that does not use the Kuhn-Tucker conditions in achieving the optimum solution. His technique involves partitioning of the variables into basic and non-basic ones. In each iteration, the basic variables and the given objective function are expressed in terms of the non-basic variables. Convergence of Beal's method is also discussed.

Both of the above methods are illustrated through examples.

8. Exercise

(i) Describe Wolfe's modified simplex method to solve a quadratic programming problem.

(ii) Use Wolfe's method in solving the quadratic programming problems given below:

a) Maximize $z = 4x_1 + 2x_2 - x_1^2 - x_2^2 - 5$

subject to $x_1 + x_2 \le 4$ and $x_1, x_2 \ge 0$.

b) Minimize $z = -x_1 - x_2 - x_3 + \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right)$

subject to $x_1 + x_2 + x_3 \le 1$.

$$4x_1 + 2x_2 \le \frac{7}{3}$$

and $x_1, x_2, x_2 \ge 0$

c) Minimize $z = x_1^2 + x_2^2 + x_3^2$

subject to $x_1 + x_2 + 3x_3 = 2$

$$5x_1 + 2x_2 + x_3 = 5$$

$$x_1, x_2, x_3 \geq 0.$$

d) Maximize $z = 8x_1 + 10x_2 - x_1^2 - x_2^2$

subject to $3x_1 + 2x_2 \leq 6$ and $x_1, x_2 \geq 0$.

e) Minimize $z = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right) + x_1 - 2x_3$

subject to $x_1 - x_2 + x_3 = 1$ and $x_1, x_2, x_3 \geq 0$.

(iii) Describe briefly the Beale's method for solving quadratic programming problem.

(iv) Solve the following quadratic programming problems by Beale's method:

a) Maximize $z = 6x_1 + 3x_2 - x_1^2 + 4x_1x_2 - 4x_2^2$

subject to the constraints :

$$x_1 + x_2 \leq 3, 4x_1 + x_2 \leq 9, x_1, x_2 \geq 0$$

b) Minimize $z = 4x_1^2 + 3x_2^2$

subject to $x_1 + 3x_2 \geq 5, x_1 + 4x_2 \geq 4; x_1, x_2 \geq 0$.

c) Maximize $z = \frac{1}{4}(2x_3 - x_1) - \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right)$

subject to the contraints :

$$x_1 - x_2 + x_3 = 1 \text{ and } x_1, x_2, x_3 \geq 0.$$

d) Maximize $z = 2x_1 + 2x_2 - 2x_2^2$

subject to $x_1 + 4x_2 \leq 4, x_1 + x_2 \leq 2$ and $x_1, x_2 \geq 0$.

v) Discuss any one method for solving a quadratic programming problem and solve :

Minimize $x_1^2 + x_2^2 - 4x_1 - 2x_2 + 5$

subject to $x_1 + x_2 \leq 4$ and $x_1, x_2 \geq 0$.

vi) Consider the problem :

Minimize $z = -4x_1 + x_1^2 - 2x_1x_2 + x_2^2$

subject to the constraints $2x_1 + x_2 \leq 6, x_1 - 4x_2 \leq 0$ and $x_1, x_2 \geq 0$.

Show that z is strictly convex and then solve the problem by any one of the quadratic programming techniques.

(vii) Solve the following non-linear programming problem using any method:

a) Minimize $f(x_1, x_2) = x_1^2 + x_2^2 - 2x_1 - 4x_2$

subject to the constraints :

$$x_1 + 4x_2 \leq 5, 2x_1 + 3x_2 \leq 6; x_1, x_2 \geq 0$$

b) Minimize $f(x_1, x_2) = 10x_1^2 + x_2^2 + 4x_1x_2 - 10x_1 - 25x_2$

subject to $x_1 + 2x_2 \leq 10, x_1 + x_2 \leq 9$ and $x_1, x_2 \geq 0$.

9. References:

The following books may be consulted as study guides.

I. Operations Research - Kanti Swarup, P.K. Gupta and Man Mohan.

II. Operations Research - S.D. Sharma.

III. Optimization - Theory and Applications - S.S. Rao.

IV. Non-linear Mathematics - T.L. Saaty and J. Bram.

V. Non-linear Programming for Operations Research - D.M. Simmons.

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-II
Paper-IX (Special Paper)
Operations Research

Module No. - 105

(ONE DIMENSIONAL MINIMIZATION METHODS)

CONTENT :

105.1 Introduction
105.2 Objective
105.3 Numerical Methods of Optimization
105.4 One Dimensional Minimization Method
105.5 Fibonacci Method
105.6 Illustrative Examples
105.7 Golden Section
105.8 Golden Section Method
105.9 Illustrative Examples
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105.11 Self Assessment Questions
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105.1 INTRODUCTION

The classical method of optimization can be used to solve the problem if the expressions for the objective function and the constraints are simple. But we cannot use classical analytical method if the optimization problem

involves the objective function and/or the constraints which are not stated as explicit functions or which are too complicated to manipulate. In such cases numerical methods of optimization are used for solving the problem.

105.2 OBJECTIVE

To solve a problem using numerical method, one dimensional minimization methods plays an important role. In numerical methods we are to minimize $f(X_i + \lambda_i S_i)$ with respect to λ_i for known X_i and S_i. This is nothing but a one dimensional minimization problem. Among many one-dimensional minimization methods Fibonacci method and Golden section methods are simple and important. These two methods are discussed here in details. Examples are also given to illustrate the methods.

105.3 Numerical Methods of Optimization

The basic philosophy of the numerical methods of optimization is to produce a sequence of approximations to the optimum point according to the following scheme.

i) Start with an initial trial point X_1

ii) Find a suitable direction S_i ($i = 1$ to start with) which points in the general direction of the minimum.

iii) Find an appropriate step length λ_i^* for movement along the direction S_i.

iv) Obtain the new approximation X_{i+1} as

$$X_{i+1} = X_i + \lambda_i^* S_i$$

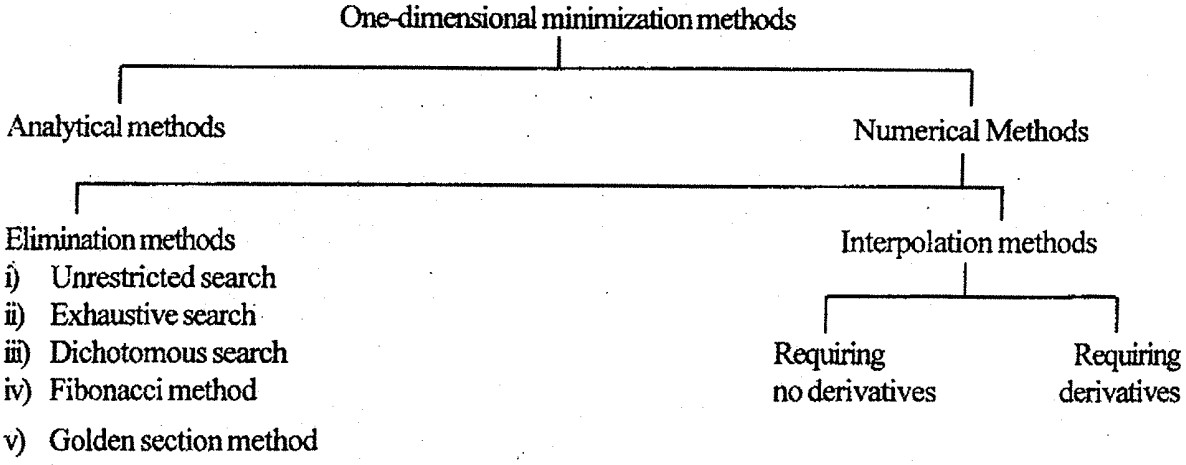
v) Test whether X_{i+1} is optimum. If X_{i+1} is optimum, stop the procedure. Otherwise, set new $i = i+1$ and repeat step (ii) onwards.

This iterative scheme is valid for unconstrained as well as constrained optimization problems.

105.4 ONE DIMENSIONAL MINIMIZATION METHODS

In numerical methods of optimization for solving the problem we have to determine step length λ_i^* for movement along the search direction S_i. This step length λ_i^* is the value of λ_i which minimizes $f(X_i + \lambda_i S_i)$ for fixed values of X_i and S_i. We note that as X_i and S_i are fixed known values $f(X_i + \lambda_i S_i)$ is nothing but a function λ_i only. Hence the problem of finding λ_i^* boils down to a problem of minimization of one variable λ_i. This is why one-dimensional minimization method is so important.

Several methods are there for solving one-dimensional minimization problem. They are as follows



105.4.1: Comparison between analytical method and numerical methods of optimization

In analytical methods the optimal value of the objective function is calculated after determining the optimal values of the decision variables. In the numerical methods of optimization, an opposite procedure is followed. In this method the values of the objective function are first found at various combinations of the decision variables and conclusions are then drawn regarding the optimal solution.

105.4.2Unimodal Function

In the process of finding optimal point often it becomes necessary that the function has only one optimum point in the domain of search. As in many methods we need only the values of the functions at various points, the function may not be continuous and differentiable. What we need is that it should be unimodal. Unimodality of a function of one variable is defined as follows.

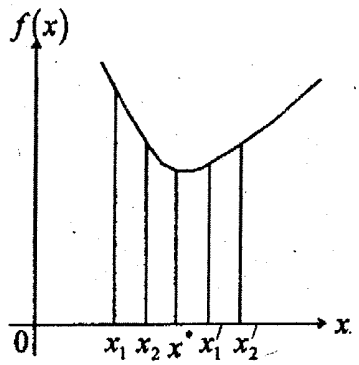
105.4.3 Definition

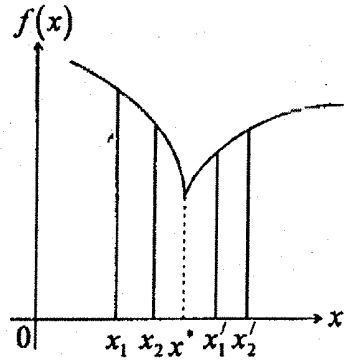
A real valued function $f(x)$ is said to be unimodal (minimum) in $[a, b]$ if there is a point $x^* \in [a,b]$ such that

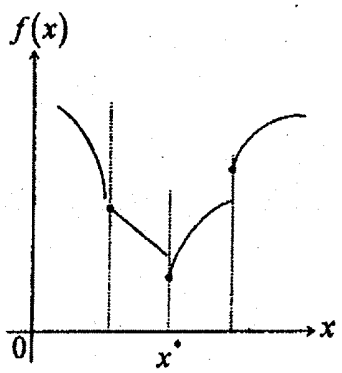
i) if $a < x_1 < x_2 < x^*$ then $f(x_1) > f(x_2)$

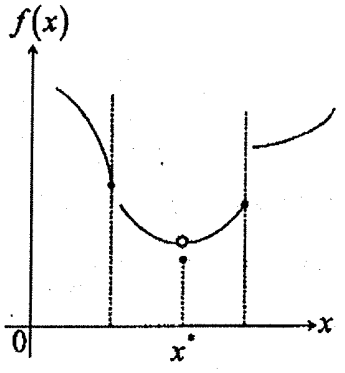
ii) if $x^* < x_1' < x_2' < b$ then $f(x_2') > f(x_1')$

we note that x^* is the minimal point of $f(x)$ and if two points are on the same side of the minimal point then the one nearer to the minimal point gives a lower value of $f(x)$ than the one further away. Also we note that for a function to be unimodal, it is not necessary to be continuous or differentiable.







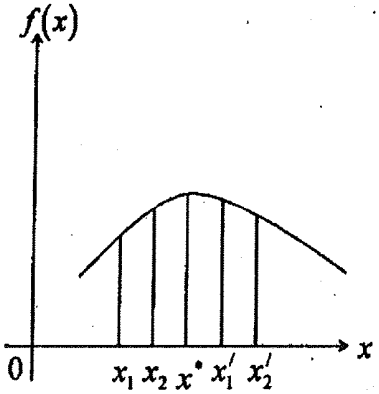


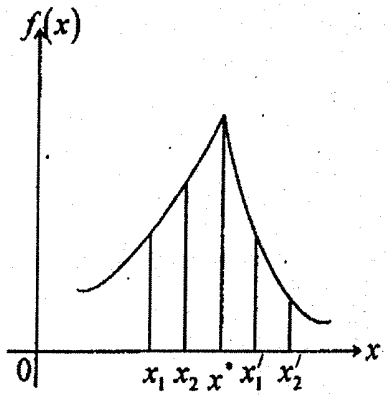
Unimodal functions (minimum)

Similarly, a real valued function $f(x)$ is said to be unimodal (maxi) in $[a,b]$ if there is a point $x^* \in [a,b]$ such that

i) if $a < x_1 < x_2 < x^*$ then $f(x_1) < f(x_2)$

ii) if $x^* < x_1' < x_2' < b$ then $f(x_2') < f(x_1')$

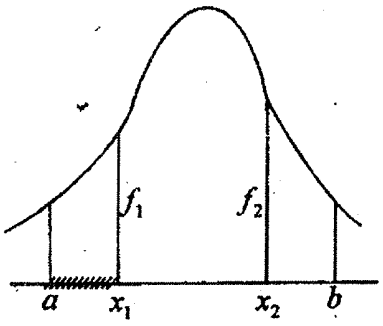


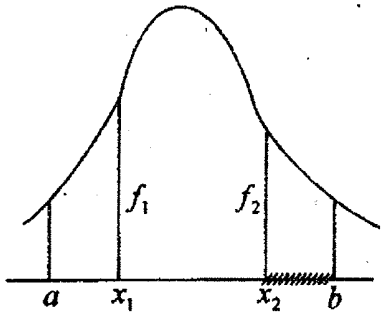


Unimodal functions (maximum)

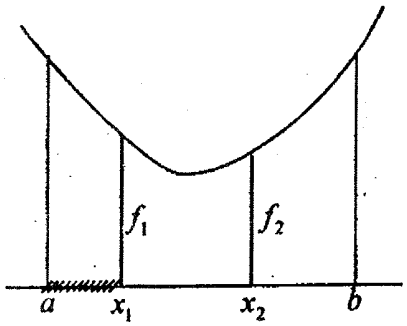
105.4.4 Elimination technique for unimodal function

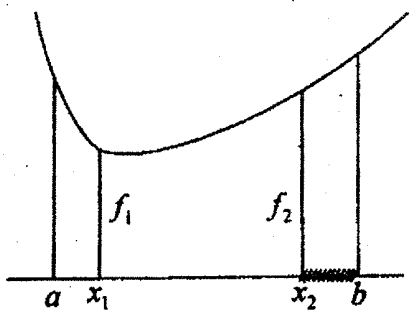
Let us consider a maximization problem for the unimodal function $f(x)$ in the interval $[a,b]$. Let $a < x_1 < x_2 < b$ and $f_1 = f(x_1), f_2 = f(x_2)$. If $f_1 < f_2$ then the maximum point cannot lie in the interval $[a,x_1]$ because of unimodality. So we can delete the interval $[a,x_1]$ and the interval of search reduces to $[x_1,b]$. On the





otherhand if $f_1 > f_2$ then the maximum point cannot lie in the interval $[x_2,b]$ because of unimodality. Thus the interval $[x_2,b]$ can be deleted and the interval of search reduces to $[a,x_2]$.





Similarly, for minimization problem if $f_1 > f_2$ then using unimodality property we can delete the interval $[a,x_1]$ and the interval of search reduces to $[x_1,b]$. On the otherhand, if $f_1 < f_2$ then the interval $[x_2,b]$ is deleted and the interval of search reduces to $[a,x_2]$.

105.5 FIBONACCI METHOD

Fibonacci method is based on Fibonacci sequence $\{F_n\}$ defined by

$$F_o = F_1 = 1$$

$$F_n = F_{n-1} + F_{n-2},\ n = 2,3,4,\ldots$$

Thus $F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_5 = 8, F_6 = 13, F_7 = 21, \ldots..$

$F_8 = 34, F_9 = 55, F_{10} = 89, F_{11} = 144, \ldots\ldots$

Fibonacci method can be used to find the optimum of a function of one variable. The function must be unimodal, it may or may not be continuous or differentiable.

This method has the following limitations:

i) The initial interval of uncertainty $[a,b]$, in which the optimum lies, has to be known

ii) The function to be optimized has to be unimodal in the initial interval of uncertainty

iii) The exact optimum point cannot be located by this method. Only an interval, known as the final interval of uncertainty can be obtained.

iv) The number of function evaluations to be used in the search has to be specified before hand.

The final interval of uncertainty can be made as small as we desire by making the number of function evaluations more.

Procedure: Let L_o be the length of the initial interval of uncertainty and $[a,b]$ be the initial interval of uncertainty. Therefore $L_o = b - a$.

Let n be the total number of experiments to be conducted. We define $L_2^* = \frac{F_{n-2}}{F_n} . L_0$

The first two experiments are placed at the points x_1 a x_2 which are located at a distance L_2^* from each end of L_o. The values of the function f at x_1, x_2 are evaluated as $f_1 = f(x_1)$ and $f_2 = f(x_2)$. Using unimodality assumption one of the intervals $[a, x_1]$ and $[x_2, b]$ is to be discarded. The remaining interval of uncertainty is denoted by L_2.

Then $L_2 = L_0 - L_2^*$

$$= L_0 - \frac{F_{n-2}}{F_n} . L_0$$

$$= L_0 \left(\frac{F_n - F_{n-2}}{F_n} \right)$$

$$= L_0 \left(\frac{F_{n-1} + F_{n-2} - F_{n-2}}{F_n} \right)$$

$$= \frac{F_{n-1}}{F_n} L_0$$

Now $L_2 - L_2^*$

$$= \frac{F_{n-1}}{F_n} L_0 - \frac{F_{n-2}}{F_n} L_0$$

$$= \frac{L_0}{F_n}(F_{n-1} - F_{n-2})$$

$$= \frac{L_0}{F_n}(F_{n-2} + F_{n-3} - F_{n-2})$$

$$= \frac{F_{n-3}}{F_n} L_0$$

$$\therefore \frac{L_2 - L_2^*}{L_2^*} = \frac{F_{n-3}}{F_{n-2}} < 1 \text{ i.e. } L_2 - L_2^* < L_2^*$$

Thus in the interval of uncertainty L_2 there is one point, either x_1 or x_2, whose distance from the two ends of L_2 are L_2^* and $L_2 - L_2^*$. The smaller of the two $L_2 - L_2^*$ and L_2^* is denoted by L_3^*, i.e. $L_3^* = L_2 - L_2^*$.

Now, $L_3^* = L_2 - L_2^* = \frac{F_{n-3}}{F_n} L_0$.

We now place the third experiment x_3 in L_2 so that the current two experiments are located at a distance L_3^* from each end of L_2. Again by the unimodal property we can reduce the interval of uncertainty from L_2 to L_3 given by $L_3 = L_2 - L_3^* = L_2^* = \frac{F_{n-2}}{F_n} L_0$.

$\therefore$ The interval of uncertainty at the end of 3rd experiment is given by

$$L_3 = \frac{F_{n-2}}{F_n} L_0$$

and this is obtained by discarding $L_3^* = \frac{F_{n-3}}{F_n} L_0$ continuing in this manner we have the following result in general.

The jth experiment is to be placed at a distance $L_j^* = \frac{F_{n-j}}{F_n} L_0$ from one end of L_{j-1} and the interval of uncertainty at the end of jth experiment is given by $L_j = \frac{F_{n-j+1}}{F_n} L_0$.

Taking $j=n$ we see that the nth experiment is to be placed at a distance $L_n^* = \frac{F_0}{F_n} L_0 = \frac{L_0}{F_n}$ from one end of L_{n-1} and the interval of uncertainty at the end of nth experiment is given by $L_n = \frac{F_1}{F_n} L_0 = \frac{L_0}{F_n}$

Now $L_{n-1} = \frac{F_{n-(n-1)+1}}{F_n} L_0 = \frac{F_2}{F_n} L_0 = \frac{2L_0}{F_n}$

$\therefore L_n^* = \frac{1}{2} L_{n-1}$

Therefore, the last two experiments are located at a distance $L_n^* = \frac{1}{2} L_{n-1}$ from each end of L_{n-1}. So they have the same location. To remove this difficulty we place the nth experiment very close to the remaining valid experiment in L_{n-1}. This enables us to obtain the final interval of uncertainty of length $\frac{1}{2} L_{n-1} = L_n = \frac{L_0}{F_n}$.

From $L_n = \frac{L_0}{F_n}$ we note that we can determine n for given L_n.

105.6 ILLUSTRATIVE EXAMPLES

Example 105.6.1 Maximize $f(x) = \begin{cases} 2x/3, & x \le 3 \\ 5-x & x > 3 \end{cases}$ in the interval $[1,4]$ by Fibonacci method using $n=6$.

Solution: Here number of experiment to be performed is $n=6$.

From Fibonacci sequence we have

$F_0 = F_1 = 1$

$F_2 = 2, F_3 = 3, F_4 = 5, F_5 = 8, F_6 = 13, F_7 = 21$ etc.

Here $L_0 = 4 - 1 = 3$.

$\therefore L_2^* = \frac{F_4}{F_6} L_0 = \frac{5}{13} \times 3 = 1.1538$

The first two experiments are placed at the positions x_1 and x_2 such that

$$x_1 = 1 + L_2^* = 1 + 1.1538 = 2.1538$$

& $x_2 = 4 - L_2^* = 4 - 1.1538 = 2.8462$

Now $f_1 = f(x_1) = \frac{2x_1}{3} = \frac{2 \times 2.1538}{3} = 1.4359$

and $f_2 = f(x_2) = \frac{2x_2}{3} = \frac{2 \times 2.8462}{3} = 1.8975$

Since $f_1 < f_2$, using unimodal property we delete the interval $[1, x_1)$. Thus this reduced interval of uncertainty is $(x_1, 4]$ i.e. [1.4359, 4] with x_2 inside it and near to x_1.

The third experiment is placed at the position x_3 given by

$$4 - x_3 = x_2 - x_1$$

or, $x_3 = 4 - x_2 + x_1$

$= 4 - 2.8462 + 2.1538$

$= 3.3076$

Now $f_3 = f(x_3) = 5 - x_3 = 5 - 3.3076 = 1.6924$.

Here $f_3 < f_2$. So by unimodality we delete the interval $(x_3, 4]$. The remaining interval of uncertainty becomes $[x_1, x_3]$ with x_2 inside it and near to the point x_3.

The fourth experiment is placed at x_4 given by

$$x_4 - x_1 = x_3 - x_2$$

$\therefore x_4 = x_1 + x_3 - x_2 = 2.1538 + 3.3076 - 2.8462 = 2.6152$

Now $f_4 = f(x_4) = \frac{2x_4}{3} = \frac{2 \times 2.1652}{3} = 1.7435$.

Since $f_4 < f_2$ we delete the interval $[x_1, x_4)$.

The remaining interval of uncertainty is $[x_4, x_3]$ with x_2 inside it and near to x_4.

The fifth experiment is placed at x_5 given by

$$x_3 - x_5 = x_2 - x_4$$

or, $x_5 = x_3 - x_2 + x_4 = 3.3076 - 2.8462 + 2.6152 = 3.0766$

Now $f_5 = f(x_5) = 5 - x_5 = 5 - 3.0766 = 1.9234$.

Since $f_5 > f_2$, using unimodal property we delete the interval $[x_4, x_2)$. The remaining interval of uncertainty is $[x_2, x_3]$ with x_5 inside it and near x_2.

The sixth experiment is placed at x_6 given by

$$x_3 - x_6 = x_5 - x_2 .$$

or, $x_6 = x_3 - x_5 + x_2 = 3.3076 - 3.0766 + 2.8462 = 3.0772$

Now $f_6 = f(x_6) = 5 - x_6 = 5 - 3.0772 = 1.9228$

Since $f_6 < f_5$, using unimodality we delete the interval $(x_6, x_3]$. The final interval of uncertainty is

$[x_2, x_6] = [2.8462, 3.0772]$

Here we note that if the exact calculation be carried out then we would get $x_5 = x_6$. In that situation x_6 should be selected very close to x_5. But here we see $x_5 \neq x_6$. This is due to round-off error involved in the calculation.

Example 105.6.2 : Minimize $f(x) = \begin{cases} (x^2 - 6x + 13)/4, & x \leq 4 \\ x - 2, & x > 4 \end{cases}$ in the interval [2, 5] by Fibonacci method taking $n = 5$.

Solution : Here $n = 5$ and $L_0 = 5 - 2 = 3$. The Fibonacci sequence is

$$F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_5 = 8, F_6 = 13, \ldots$$

$$\therefore L_2^* = \frac{F_3}{F_5} L_0 = \frac{3}{8} \times 3 = \frac{9}{8} = 1.125$$

The first two experiments are placed at the positions x_1 and x_2 such that

$$x_1 = 2 + L_2^* = 2 + 1.125 = 3.125$$

$$x_2 = 5 - L_2^* = 5 - 1.125 = 3.875$$

Now $f_1 = f(x_1) = (x_1^2 - 6x_1 + 13)/4 = (3.125^2 - 6 \times 3.125 + 13)/4 = 1.039$

and $f_2 = f(x_2) = (x_2^2 - 6x_2 + 13)/4 = (3.875^2 - 6 \times 3.875 + 13)/4 = 1.194$

Since $f_2 > f_1$, using unimodal property we delete the interval $(x_2, 5]$. Thus the reduced interval of uncertainty is $[2, x_2]$ with x_1 inside it close to x_2.

The third experiment is placed at the position x_3 given by

$$x_3 - 2 = x_2 - x_1 .$$

or, $x_3 = 2 + x_2 - x_1 = 2 + 3.875 - 3.125 = 2.750$

Now $f_3 = f(x_3) = (x_3^2 - 6x_3 + 13)/4 = (2.75^2 - 6 \times 2.75 + 13)/4 = 1.0156$

Since $f_3 > f_1$, by unimodality we delete the interval $[2, x_3)$. The remaining interval of uncertainty is $[x_3, x_2]$ with x_1 inside it and close to x_3.

The fourth experiment is placed at the position x_4 given by

$$x_2 - x_4 = x_1 - x_3$$

or, $$x_4 = x_2 - x_1 + x_3 = 3.875 - 3.125 + 2.750 = 3.5$$

Now $f_4 = f(x_4) = (x_4^2 - 6x_4 + 13)/4 = (3.5^2 - 6 \times 3.5 + 13)/4 = 1.0625$

Since $f_4 > f_1$, we delete the interval $(x_4, x_2]$. The reduced interval of uncertainty is $[x_3, x_4]$ with x_1 inside it and close to x_4.

The fifth experiment is placed at the position x_5 given by

$$x_5 - x_3 = x_4 - x_1$$

or, $$x_5 = x_3 + x_4 - x_1 = 2.75 + 3.5 - 3.125 = 3.125$$

We note that $x_1 = x_5$. So we take x_5 very close to x_1 as $x_5 = 3.124$.

Now $f_5 = f(x_5) = (x_5^2 - 6x_5 + 13)/4 = (3.124^2 - 6 \times 3.124 + 13)/4 = 1.003844$

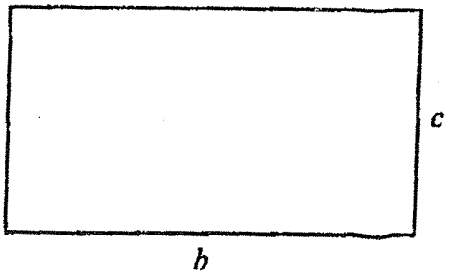
Since $f_1 > f_5$, using unimodal property we delete the interval $(x_1, x_4]$. Hence the final interval of uncertainty is $[x_3, x_1]$ i.e. $[2.75, 3.125]$.

105.7 GOLDEN SECTION

Ancient Greek architects believed that a building having sides b and c satisfying the relation

$$\frac{b+c}{b} = \frac{b}{c} = \gamma$$

will be having the most pleasing properties. This ratio is called Golden ratio. It is also found in Euclid's geometry that the division of a line segment into unequal parts so that the ratio of the whole to the largest part is equal to the ratio of the larger part to the smaller part. This section is known as the golden section.



Thus the golden section is

$$\frac{AC}{AB} = \frac{AB}{BC} = \gamma \quad \text{i.e.} \quad \frac{AB + BC}{AB} = \frac{AB}{BC} = \gamma$$

From this we have

$$\frac{AB}{AB}+\frac{BC}{AB}=\frac{AB}{BC}=\gamma$$

or, $1+\frac{1}{\gamma}=\gamma$

or, $\gamma^2-\gamma-1=0$

$$\therefore \gamma = \frac{-(-1)\pm\sqrt{(-1)^2-4.1.(-1)}}{2.1}$$

$$= \frac{1\pm\sqrt{5}}{2}$$

Since γ is positive number we have

$$\gamma = \frac{\sqrt{5}+1}{2} = 1.618$$

105.8 GOLDEN SECTION METHOD

Golden section method is similar to the Fibonacci method except for one difference. The difference is that in Fibonacci method the total number of experiments to be performed has to be specified before beginning the calculation, whereas, this is not required in golden section method. In fact when n is very very large then Fibonacci method reduces to golden section method. In Fibonacci method the number of experiments to be performed is decided at the beginning but in golden section method the total number of experiments are to be decided during the computations.

In the Fibonacci method the interval of uncertainty at the end of two experiments is given by

$$L_2 = \frac{F_{n-1}}{F_n} L_0.$$

In Golden Section method as n is very large this L_2 becomes

$$L_2 = \lim_{n\to\alpha}\frac{F_{n-1}}{F_n} L_0 = L_0\left(\lim_{n\to\alpha}\frac{F_{n-1}}{F_n}\right)$$

Also in Fibonacci method L_3 is given by

$$L_3 = \frac{F_{n-2}}{F_n} L_0$$

$\therefore$ In Golden section method L_3 will be given by

$$L_3 = \lim_{n\to\alpha} \frac{F_{n-2}}{F_n} L_0$$

$$= \lim_{n\to\alpha} \left(\frac{F_{n-2}}{F_{n-1}} \cdot \frac{F_{n-1}}{F_n} L_0 \right)$$

$$= L_0 \cdot \left(\lim_{n\to\alpha} \frac{F_{n-1}}{F_{n-1}} \right) \left(\lim_{n\to\alpha} \frac{F_{n-1}}{F_n} \right)$$

$$= L_0 \cdot \left(\lim_{n\to\alpha} \frac{F_{n-2}}{F_n} \right) \left(\lim_{n\to\alpha} \frac{F_{n-1}}{F_n} \right)$$

$$= L_0 \left(\lim_{n\to\alpha} \frac{F_{n-1}}{F_n} \right)^2$$

Similarly, we get $L_4 = L_0 \left(\lim_{n\to\alpha} \frac{F_{n-1}}{F_n} \right)^3$.

Generalizing these results we have

$$L_k = \left(\lim_{n\to\alpha} \frac{F_{n-1}}{F_n} \right)^{k-1} . L_0$$

We have the relation

$$F_n = F_{n-1} + F_{n-2}$$

$$\therefore \frac{F_n}{F_{n-1}} = 1 + \frac{F_{n-2}}{F_{n-1}}$$

$$\text{or, } \lim_{n\to\alpha} \frac{F_n}{F_{n-1}} = 1 + \lim_{n\to\alpha} \frac{F_{n-2}}{F_{n-1}}$$

$$= 1 + \lim_{n\to\alpha} \frac{1}{\frac{F_{n-1}}{F_{n-2}}}$$

$$= 1 + \frac{1}{\lim_{n\to\alpha} \frac{F_{n-1}}{F_{n-2}}}$$

$$= 1 + \frac{1}{\lim_{n\to\alpha} \frac{F_n}{F_{n-1}}}$$

Let $\gamma = \lim_{n\to\alpha} \frac{F_n}{F_{n-1}}$

$\therefore$ We have $\gamma = 1 + \frac{1}{\gamma}$

or, $\gamma^2 = \gamma + 1$

or, $\gamma^2 - \gamma - 1 = 0$

or, $\gamma = \frac{-(-1) \pm \sqrt{(-1)^2 - 4.1(-1)}}{2.1}$

$$= \frac{1 \pm \sqrt{5}}{2}$$

Since γ is a positive real number, we have

$\gamma = \frac{\sqrt{5}+1}{2} = 1.618$, which is nothing but golden ratio or golden section.

Hence we have, in general,

$$L_k = \left(\frac{1}{\gamma}\right)^{k-1} L_0 = (0.618)^{k-1} L_0 .$$

$\therefore$ In the Golden section method the interval of uncertainty at the end of kth experiment is given by

$$L_k = (0.618)^{k-1} L_0 .$$

105.8.1 Procedure of Golden Section Method

In the Fibonacci method, the location of the first two experiments are the point situated at a distance L_2^* from the two ends of the initial interval of uncertainty, where L_2^* is given by

$$L_2^* = \frac{F_{n-2}}{F_n} L_0 .$$

In Golden Section method n is very large. Therefore L_2^* is given by

$$L_2^* = \lim_{n\to\alpha} \frac{F_{n-2}}{F_n} L_0$$

$$= \lim_{n\to\alpha} \left(\frac{F_{n-2}}{F_{n-1}} \cdot \frac{F_{n-1}}{F_n} L_0 \right)$$

$$= L_0 . \lim_{n\to\alpha} \frac{F_{n-2}}{F_{n-1}} . \lim_{n\to\alpha} \frac{F_{n-1}}{F_n}$$

$$= L_0 . \lim_{n\to\alpha} \frac{F_{n-1}}{F_n} . \lim_{n\to\alpha} \frac{F_{n-1}}{F_n}$$

$$= L_0 . \left(\lim_{n\to\alpha} \frac{F_{n-1}}{F_n} \right)^2$$

$$= L_0 \left(\frac{1}{\gamma} \right)^2$$

$$= L_0 (.613)^2 = 0.382 L_0 .$$

$\therefore$ In Golden section method, the first two experiments are placed at the points x_1 and x_2 which are located at a distance $L_2^* = 0.382 L_0$ from each end of L_0. The values of the functions f at x_1, x_2 are evaluated as $f_1 = f(x_1)$ and $f_2 = f(x_2)$. Using the assumption of unimodality, one of the two intervals $[a, x_1]$ and $[x_2, b]$ can be discarded. The remaining interval of uncertainty will be $L_2 = 0.618\, L_2$. This interval will contain one experiment point. The smaller distance of this experiment point from the ends of L_2 is denoted by L_3^*. The third experiment x_3 is placed in L_2 so that the current two experiments are located at a distance L_3^* from each end of L_2. Again using unimodality we can discard one of the end intervals and the reduced interval of uncertainty at the end of 3rd experiment becomes $L_3 = (0.618)^2 L_0$. This process is continued until the desired length of the interval of uncertainty is obtained.

105.9 ILLUSTRATIVE EXAMPLES

Example 105.9.1: Maximize $f(x) = \begin{cases} 2x/3, & x \le 3 \\ 5 - x, & x > 3 \end{cases}$ in the interval [1,4] by Golden Section method upto six experiments.

Solution: We have $L_0 = 4 - 1 = 3$.

Now $L_2^* = .382\ L_0 = .382 \times 3 = 1.146$.

The first two experiments are placed at the positions x_1 and x_2 such that

$$x_1 = 1 + L_2^* = 1 + 1.146 = 2.146$$

$$x_2 = 4 - L_2^* = 4 - 1.146 = 2.854$$

Now $f_1 = f(x_1) = \frac{2x_1}{3} = \frac{2 \times 2.146}{3} = 1.43066$

$$f_2 = f(x_2) = \frac{2x_2}{3} = \frac{2 \times 2.854}{3} = 1.90266$$

As $f_1 < f_2$ and the problem is of maximization, using unimodal property we delete the interval $[1, x_1)$. Thus the reduced interval of uncertainty is $[x_1, 4]$ with x_2 inside it and near to the point x_1.

The third experiment is to be placed at x_3 given by

$$4 - x_3 = x_2 - x_1$$

or, $x_3 = 4 - x_2 + x_1 = 4 - 2.854 + 2.146 = 3.292.$

Now $f_3 = f(x_3) = 5 - 3.292 = 1.708$.

Here $f_3 < f_2$. So by unimodality we delete the interval $(x_3, 4]$. The remaining interval of uncertainty becomes $[x_1, x_3]$ with x_2 inside it and near to the point x_3.

The fourth experiment is placed at x_4 given by

$$x_4 - x_1 = x_3 - x_2$$

or, $x_4 = x_1 + x_3 - x_2 = 2.146 + 3.292 - 2.854 = 2.584.$

Now $f_4 = f(x_4) = \frac{2x_4}{3} = \frac{2 \times 2.584}{3} = 1.7226$

Here $f_4 < f_2$. Using unimodality we delete the interval $[x_1, x_4)$. The remaining interval of uncertainty is $[x_4, x_3]$ with x_2 inside it and near to x_4.

The fifth experiment is placed at x_5 given by

$$x_3 - x_5 = x_2 - x_4$$

or, $x_5 = x_3 - x_2 + x_4 = 3.292 - 2.854 + 2.584 = 3.022$

Now $f_5 = f(x_4) = 5 - 3.022 = 1.978$.

Since $f_5 > f_2$, using unimodal property we delete the interval $[x_4, x_2)$. The remaining interval of uncertainty is $[x_2, x_3]$ with x_5 inside it and near to x_2.

The sixth experiment is placed at x_6 given by

$$x_3 - x_6 = x_5 - x_2$$

or, $x_6 = x_3 - x_5 + x_2 = 3.292 - 3.022 + 2.854 = 3.124$

Now $f_6 = f(x_6) = 5 - 3.124 = 1.876$

Since $f_6 < f_5$, using unimodality we delete the interval $(x_6, x_3]$. The final interval of uncertainty is given by $[x_2, x_6]$ i.e. [2.854, 3.124].

Example 105.9.2: Minimize $f(x) = \begin{cases} (x^2 - 6x + 13)/4, & x \le 4 \\ x - 2, & x > 4 \end{cases}$ in the interval [2, 5] by golden section method.

Solution : Here $L_0 = 5 - 2 = 3$

$$\therefore L_2^* = .382 \times L_0 = .382 \times 3 = 1.146$$

The first two experiments are placed at the positions x_1 and x_2 given by

$$x_1 = 2 + 1.146 = 3.146$$

$$x_2 = 5 - 1.146 = 3.854$$

Now $f_1 = f(x_1) = (x_1^2 - 6x_1 + 13)/4 = (3.146^2 - 6 \times 3.146 + 13)/4$

$= (9.8973 - 18.876 + 13) / 4 = 1.005329$

$f_2 = f(x_2) = (x_2^2 - 6x_2 + 13)/4 = (3.854^2 - 6 \times 3.854 + 13)/4$

$= (14.8533 - 23.124 + 13) / 4 = 1.182329$

As $f_1 < f_2$ and the problem is of minimization, we delete the interval $(x_2, 5]$. Thus the reduced interval of uncertainty is $[2, x_2]$ with x_1 inside it and close to the point x_2.

The third experiment is to be placed at x_3 given by

$$x_3 - 2 = x_2 - x_1$$

or, $x_3 = 2 + x_2 - x_1 = 2 + 3.854 - 3.146 = 2.708$

Now $f_3 = f(x_3) = (x_3^2 - 6x_3 + 16)/4 = (2.708^2 - 6 \times 2.708 + 13)/4$

$= (7.33326 - 16.248 + 13) / 4 = 1.021316$

As $f_1 < f_3$, the interval $[2, x_3)$ is deleted by unimodality. The remaining interval of uncertainty is $[x_3, x_2]$ with x_1 inside it and nearer to x_3.

The fourth experiment is placed at x_4 given by

$$x_2 - x_4 = x_1 - x_3$$

or, $x_4 = x_2 - x_1 + x_3 = 3.854 - 3.146 + 2.708 = 3.416$

Now $f_4 = f(x_4) = (x_4^2 - 6x_4 + 13)/4 = (3.416^2 - 6 \times 3.416 + 13)/4$

$= (11.6691 - 20.496 + 13) / 4 = 1.043264$

As $f_1 < f_4$, by unimodality we delete the interval $(x_4, x_2]$. The interval of uncertainty after reduction is thus $[x_3, x_4]$ with x_1 inside it and nearer to x_4.

The fifth experiment is placed at x_5 given by

$$x_5 - x_3 = x_4 - x_1$$

or, $x_5 = x_3 + x_4 - x_1 = 2.708 + 3.416 - 3.146 = 2.978$

Now $f_5 = f(x_5) = (x_5^2 - 6x_5 + 13)/4 = (2.978^2 - 6 \times 2.978 + 13)/4$

$= (8.8685 - 17.868 + 13)/4 = 1.00012$

As $f_1 > f_5$, using unimodality we delete the interval $(x_1, x_4]$. The reduced interval of uncertainty is $[x_3, x_1]$ with x_5 inside it and nearer to the point x_1.

The sixth experiment is placed at x_6 such that

$$x_6 - x_3 = x_1 - x_5$$

or, $x_6 = x_3 + x_1 - x_5 = 2.708 + 3.146 - 2.978 = 2.876$

Now $f_6 = f(x_6) = (x_6^2 - 6x_6 + 13)/4 = (2.876^2 - 6 \times 2.876 + 13)/4$

$= (8.27137 - 17.256 + 13)/4 = 1.003844$

Using unimodal property, we delete the interval $[x_3, x_6)$. The reduced interval of uncertainty is $[x_6, x_1]$ i.e. [2.876, 3.146].

105.10 SUMMARY

The necessity of numerical methods of optimization is discussed. The importance of one-dimensional minimization methods in solving multivariable optimization problems is described. The concept of unimodal function and its role in the elimination methods is presented. Fibonacci method and Golden section methods are discussed in detail through examples.

105.11 SELF ASSESSMENT QUESTIONS

1. Maximize $f(x)=\begin{cases}3x/2, & x\le 2\\ 5-x, & x>2\end{cases}$ in the interval [0, 5] by Fibonacci method using $n=5$

2. Minimize $f(x)=\begin{cases}8-x, & x\le 4\\ x, & x>4\end{cases}$ in the interval [1, 7] by Fibonacci method using $n=6$

3. Minimize $f(x)=|x-1|$ in the interval [–1, 5] by Fibonacci method using $n=5$

4. Maximize $f(x)=\begin{cases}2\sqrt{x}, & x\le 1\\ 3-x, & x>1\end{cases}$ in the interval [0, 4] by Fibonacci method using $n=4$

5. Maximize $f(x)=\begin{cases}4x/3, & x\le 3\\ 7-x, & x>3\end{cases}$ in the interval [1, 5] by Golden section method upto six experiments

6. Minimize $f(x)=\begin{cases}6-x, & x\le 5\\ 2x-9, & x>5\end{cases}$ in the interval [2, 8] by Golden section method upto five experiments

7. Maximize $f(x)=\begin{cases}2\sqrt{x}, & x\le 1\\ 3-x, & x>1\end{cases}$ in the interval [0, 5] by Golden section method upto six experiments

8. Minimize $f(x)=|x|$ in the interval [–2, 2] by Golden section method upto six experiments

9. Maximize $f(x)=\begin{cases}3x/2, & x\le 2\\ 5-x, & x>2\end{cases}$ in the interval [0, 6] by Golden section method upto five experiments

105.12 SUGGESTED FURTHER READINGS

1. Optimization Theory and Applications : S.S. Rao, Wiley Eastern Limited.
2. Optimization methods in operations research and sysetems analysis : K.V. Mital & C. Mohan, New Age International Publishers.

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-II
Paper - IX (Special Paper)
Operations Research
Module No. - 106
(Unconstrained and Constrained Optimization Technique)

CONTENT :

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Unconstrained Optimization Technique

106.1 Introduction

The characteristic of unconstrained programming problem is that the solution vector $(x_1, x_2, ..., x_n)$ need not satisfy any constraint. The study of such programming problem is important because of the following reasons:

i) Some of the most powerful and convenient methods of solving constrained optimization problems involve the transformation of the problem into one of unconstrained optimization.

ii) The study of the unconstrained optimization methods provides the basic understanding necessary for the study of the constrained optimization methods.

Several methods are available for solving an unconstrained optimization problem. These methods are classified into two broad categories viz. direct search methods and descent methods. The different methods of these two categories are shown below.

Direct search methods

i) Random search method

ii) Univariate method

iii) Pattern search method

iv) Rosenbrock's method

v) Simplex method

Descent methods

i) Steepest descent method

ii) Conjugate gradient method

iii) Davidon-Fletcher-Powell method

iv) Newton's method.

The direct search methods require only objective function evaluations and do not use the partial derivatives of the function in finding the optimum. Hence they are often called the nongradient methods. These methods are most suitable for simple problems involving a relatively small number of variables. These methods are, in general, less efficient than descent methods.

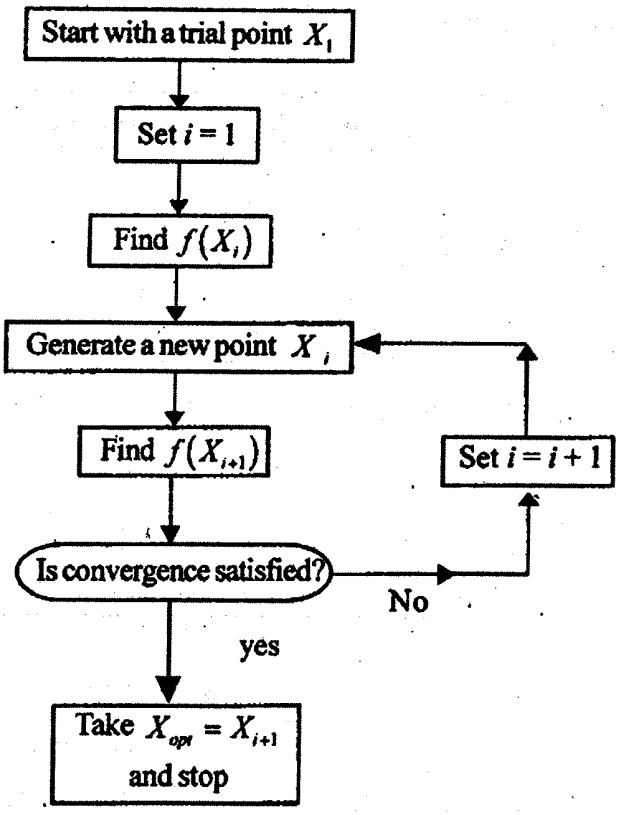
The descent methods require the evaluation of the objective function as well as its first and higher order derivatives. The descent methods are generally more efficient compared to the direct search methods because more informations about the function to be optimized are available through the use of derivatives. The descent methods are also known as gradient methods.

106.2 Objective

In reality often we face the problems of optimization, both unconstrained and constrained types. Due to the complex nature of these problems often analytical methods are not applicable to these problems, and so we are to depend on numerical methods. In numerical method we are to choose some point to start with and find the value of the function there. Then we are to take a tentative short step in some direction and find the value of the function at this new point. If the value found is seen better than the previous value, we discard the first point in favour of the second. We are to go on doing this till we find a point where the value of the function becomes optimal or lies within a tolerable limit. There are many iterative schemes of optimization. These schemes differ from one another depending on the methods of choosing direction of search and corresponding step lengths. The objective of this module is to study some of these unconstrained and constrained methods. Each such method is explained with the help of examples.

106.3 General Iterative Scheme of Optimization

All the unconstrained optimization methods are iterative in nature. Hence they start from an initial trial solution and proceed towards the optimum point in a sequential manner. It is important to note that all the unconstrained optimization methods requires an initial point X_1 to start the iterative procedure. One method differs from another only in the method of generation the new point X_{i+1} from X_i and in testing the point X_{i+1} for optimality. The general iterative scheme is shown below as a flow diagram.



If the search direction from X_i be S_i and the step length for movement along the search direction S_i be λ_i^* then the next point to X_i is obtained as $X_{i+1} = X_i + \lambda_i^* S_i$. Thus the iterative scheme becomes

i) Start with an initial trial point X_1

ii) Find a suitable direction S_i ($i = 1$ to start with) which points in general direction of the minimum.

iii) Find an appropriate step length λ_i^* for movement along the direction S_i

iv) Obtain the new approximation X_{i+1} as

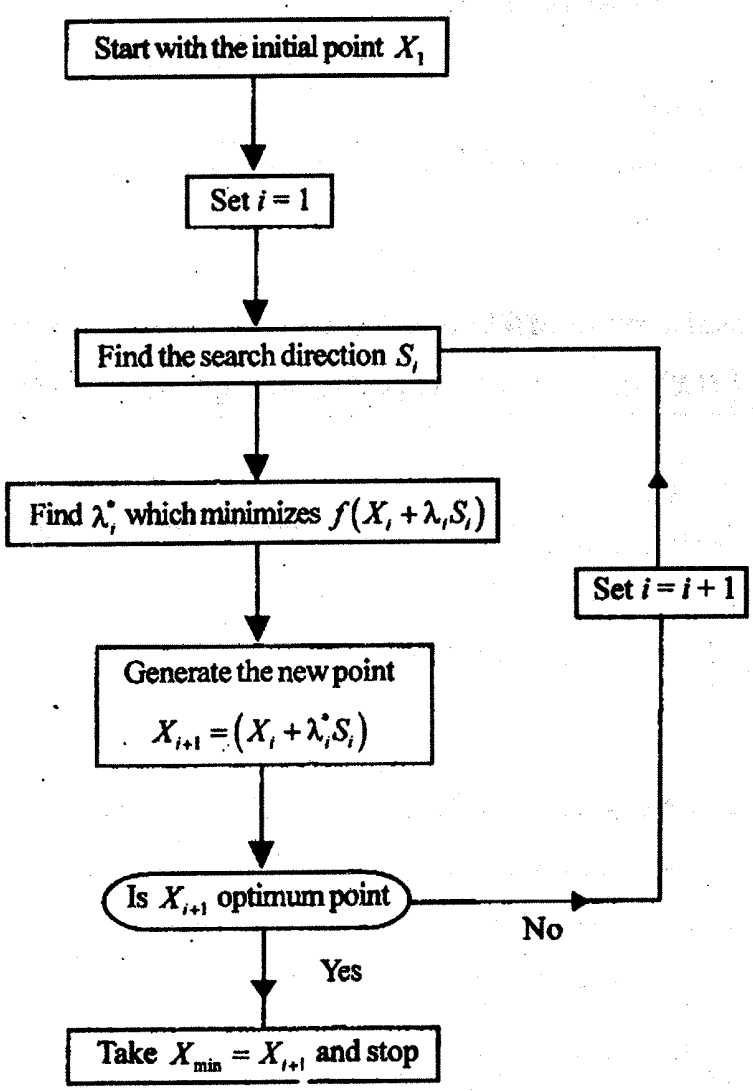
$$X_{i+1} = X_i + \lambda_i^* S_i.$$

v) Test whether X_{i+1} is optimum. If X_{i+1} is optimum then stop the procedure, otherwise set new $i = i + 1$ and repeat step (ii) onward.

Thus as mentioned before, the efficiency of an optimization method depends on the efficiency with which the quantities λ_i^* and S_i are determined to generate the new point X_{i+1} as $X_i + \lambda_i^* S_i$. To find λ_i^* we are to minimize $f(X_i + \lambda_i S_i)$ regarding it as a function of λ_i only.

$$f(X_i + \lambda_i^* S_i) = \min_{\lambda_i} \{f(X_i + \lambda_i S_i)\}$$

The flow chart for the iterative scheme may thus be shown as follows :



106.4 Steepest Descent Method

In the steepest descent method to minimize a function f of n variables $x_1, x_2, \ldots, x_n$ we use the gradient of the function f defined by

$$\nabla f = \left[\frac{\partial f}{\partial x_1} \frac{\partial f}{\partial x_2} \ldots \frac{\partial f}{\partial x_n}\right]^T$$

The gradient of f is a n-component vector and has a very important property viz. if we move along the

gradient direction from any point in the n-dimensional space then the function value increases at the fastest rate. To prove this property we first define directional derivative.

Definition

Directional Derivative :

The directional derivative of $f(X)$ in the direction of the unit vector y is defined as the following limit :

$$\lim_{t\to 0}\frac{f(X+tY)-f(X)}{t}$$

The directional derivative of $f(X)$ in the direction Y thus given by using Taylor's theorem

$$\lim_{t\to 0}\frac{\{f(X)+(tY)^1\nabla f(X)+\text{terms of higher degree in } t\}-f(X)}{t}$$

$$= Y'\nabla f(X).$$

$\therefore$ The directional derivative of $f(X)$ in the direction of unit vector Y

$= Y'\nabla f(X).$

= rate of change of $f(X)$ in the direction of Y.

Theorem 106.4.1

Prove that $f(X)$ increases at the fastest rate in the direction of ∇f.

Proof. We have that the rate of change of $f(X)$ in the direction of the unit vector Y is $Y'\nabla f(X)$ ..... (1)

Now the unit vector in the direction of the gradient vector ∇f is $\nabla f / |\nabla f|$. Therefore, the rate of change of $f(X)$ in the direction of the gradient vector

$$= \left(\frac{\nabla f}{|\nabla f|}\right)'\nabla f = \frac{(\nabla f)'(\nabla f)}{|\nabla f|} = \frac{|\nabla f|^2}{|\nabla f|} = |\nabla f| \quad \text{..... (2)}$$

Since $|\nabla f| > 0$, it follows that $f(X)$ increases in the direction of ∇f

Using Cauchy Schwarz inequality we have

$$|Y'\nabla f| \le |Y||\nabla f| = |\nabla f| \quad [\because |Y| = 1] \quad \text{..... (3)}$$

From (1), (2) and (3) it follows that the rate of change of $f(X)$ in the direction of ∇f is greater than that in the direction of any unit vector Y. In other words $f(X)$ increases at the fastest rate in the direction of ∇f.

Note : Since $f(X)$ increases at the fastest rate in the direction of ∇f, it follows that $f(X)$ decreases at the fastest rate in the direction of $-\nabla f$. Thus the direction of ∇f and $-\nabla f$ are respectively the directions of the steepest ascent and steepest descent.

106.4.2 Iterative Scheme of Steepest Descent Method

The steepest descent method uses the property that a function $f(X)$ decreases at the fastest rate in the direction of $-\nabla f$. Thus at X_i the function decreases at the fastest rate along the direction S_i given by $S_i = [-\nabla f]_{X_i} = -\nabla f_i$.

The iterative scheme of steepest descent method is given below:

i) Start with an initial point X_1

ii) Take the search direction S_i at X_i ($i = 1$ to start with) as $S_i = [-\nabla f]_{X_i}$ and denote it by $-\nabla f_i$

iii) Find the step length λ_i^* for movement along S_i which minimizes $f(X_i + \lambda_i S_i)$

iv) Obtain the new approximation point X_{i+1} as

$$X_{i+1} = X_i + \lambda_i^* S_i$$

v) Test whether X_{i+1} is optimum. If X_{i+1} is optimum then stop the procedure. Otherwise set new $i = i + 1$ and repeat step (ii) onward.

106.5 ILLUSTRATIVE EXAMPLES

Example 105.5.1: Using steepest descent method minimize $f = x_1^2 + x_2^2 + 2gx_1 + 2fy_1 + c$ starting from the point $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$.

Solution : Here $f(x_1, x_2) = x_1^1 + x_2^2 + 2gx_1 + 2fy_1 + c$

$\therefore$ The gradient of f is given by

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 + 2g \\ 2x_2 + 2f \end{bmatrix}$$

The starting point is $X_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$. Using steepest descent method the search direction at X_1 is given by

$$S_1 = [-\nabla f]_{X_1} = \begin{bmatrix} -2\alpha - 2g \\ -2\beta - 2f \end{bmatrix}$$

The step length λ_1^* is obtained by minimising $f(X_1 + \lambda_1 S_1)$ with respect to λ_1.

Now $(X_1 + \lambda_1 S_1) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \lambda_1 \begin{bmatrix} -2\alpha - 2g \\ -2\beta - 2f \end{bmatrix} = \begin{bmatrix} \alpha - 2\lambda_1\alpha - 2\lambda_1 g \\ \beta - 2\lambda_1\beta - 2\lambda_1 f \end{bmatrix} = \begin{bmatrix} \gamma \\ \delta \end{bmatrix}$

where $\gamma = \alpha + \lambda_1(-2\alpha - 2g)$

and $\delta = \beta + \lambda_1(-2\beta - 2f)$

$\therefore f(X_1 + \lambda_1 S_1) = \gamma^2 + \delta^2 + 2g\gamma + 2f\delta + C$

For minimum value of f we have $\frac{df}{d\lambda_1} = 0$. This gives $\frac{\partial f}{\partial \gamma}\frac{\partial \gamma}{\partial \lambda_1} + \frac{\partial f}{\partial \delta}\frac{\partial \delta}{\partial \lambda_1} = 0$

or, $(2\gamma + 2g)(-2\alpha - 2g) + (2\delta + 2f)(-2\beta - 2f) = 0$

or, $(\gamma + g)(\alpha + g) + (\delta + f)(\beta + f) = 0$

or, $\{\alpha + \lambda_1(-2\alpha - 2g) + g\}(\alpha + g) + \{\beta + \lambda_1(-2\beta - 2f) + f\}(\beta + f) = 0$

or, $(\alpha + g)^2 - 2\lambda_1(\alpha + g)^2 + (\beta + f)^2 - 2\lambda_1(\beta + f)^2 = 0$

or, $(1 - 2\lambda_1)\left[(\alpha + g)^2 + (\beta + f)^2\right] = 0$

or, $1 - 2\lambda_1 = 0$ or, $\lambda_1 = \frac{1}{2}$

$\therefore \lambda_1^* = \frac{1}{2}$.

Now, X_2 is given by $X_2 = X_1 + \lambda_1^* S_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -2\alpha - 2g \\ -2\beta - 2f \end{bmatrix}$

$$= \begin{bmatrix} \alpha - \alpha - g \\ \beta - \beta - f \end{bmatrix} = \begin{bmatrix} -g \\ -f \end{bmatrix}$$

The gradient of f at X_2 is given by

$$[\nabla f]_{X_2} = \begin{bmatrix} 2(-g) + 2g \\ 2(-f) + 2f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This shows that X_2 is the optimum point.

$$\therefore X_{opt} = X_2 = \begin{bmatrix} -g \\ -f \end{bmatrix}$$

Example 105.5.2 : Using steepest descent method minimize $f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$ starting from the point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Solution : Here $f = f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$ and the starting point is $X_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The gradient of f is given by

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 + 4x_1 + 2x_2 \\ -1 + 2x_1 + 2x_2 \end{bmatrix}$$

$$\therefore \nabla f_1 = [\nabla f]_{X_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The search direction at X_1 is given by

$$S_1 = -\nabla f_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

To find X_2 we are to find the optimal step length λ_1^*. For this we are to minimize $f(X_1 + \lambda_1 S_1)$ with respect to λ_1.

Now $X_1 + \lambda_1 S_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\lambda_1 \\ \lambda_1 \end{bmatrix}$.

$$\therefore f(X_1 + \lambda_1 S_1) = -\lambda_1 - \lambda_1 + 2\lambda_1^2 - 2\lambda_1^2 + \lambda_1^2 = \lambda_1^2 - 2\lambda_1$$

For minimum value of f we have $\frac{df}{d\lambda_1} = 0$.

From this we have $2\lambda_1 - 2 = 0$ or, $\lambda_1 = 1$ $\therefore \lambda_1^* = 1$.

Thus we obtain X_2 as

$$X_2 = X_1 + \lambda_1^* S_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The gradient of f at X_2 is given by

$$\nabla f_2 = [\nabla f]_{X_2} = \begin{bmatrix} 1-4+2 \\ -1-2+2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore X_2$ is not an optimum point. So we proceed to the next iteration.

The search direction at X_2 is given by

$$S_2 = -\nabla f_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

To find X_3 we find the step length λ_3^* by minimizing $f(X_2 + \lambda_2 S_2)$ with respect to λ_2.

Now $X_2 + \lambda_2 S_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+\lambda_2 \\ 1+\lambda_2 \end{bmatrix}$

$$\therefore f(X_2 + \lambda_2 S_2) = (-1+\lambda_2) - (1+\lambda_2) + 2(-1+\lambda_2)^2 + 2(-1+\lambda_2)(1+\lambda_2) + (1+\lambda_2)^2$$
$$= -1+\lambda_2 - 1 - \lambda_2 + 2 - 4\lambda_2 + 2\lambda_2^2 - 2 + 2\lambda_2^2 + 1 + 2\lambda_2 + \lambda_2^2$$
$$= -1 - 2\lambda_2 + 5\lambda_2^2$$

To minimize f we set $\frac{df}{d\lambda_2} = 0$.

From this we have $-2 + 10\lambda_2 = 0$ or, $\lambda_2 = +\frac{1}{5}$ $\therefore \lambda_2^* = \frac{1}{5}$.

Hence $X_3 = X_2 + \lambda_2^* S_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.8 \\ 1.2 \end{bmatrix}$

The gradient of f at X_3 is given by

$$\nabla f_3 = [\nabla f]_{X_2} = \begin{bmatrix} 1+4(-0.8)+2(1.2) \\ -1+2(-0.8)+2(1.2) \end{bmatrix} = \begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore X_3$ is not optimum and we proceed to the next iteration.

The search direction at X_3 is given by

$$S_3 = -\nabla f_3 = \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix}$$

To find X_4 we are to find the step length λ_4^* by minimizing $f(X_3+\lambda_3 S_3)$ with respect to λ_3.

Now $X_3+\lambda_3 S_3=\begin{bmatrix}-0.8\\1.2\end{bmatrix}+\lambda_3\begin{bmatrix}-0.2\\0.2\end{bmatrix}=\begin{bmatrix}-0.8-\lambda_3 0.2\\1.2+\lambda_3 0.2\end{bmatrix}$

$$\therefore f(X_3+\lambda_3 S_3)=(-0.8-0.2\lambda_3)-(1.2+0.2\lambda_3)+2(0.8-0.2\lambda_3)^2$$
$$+2(-0.8-0.2\lambda_3)(1.2+0.2\lambda_3)+(1.2+0.2\lambda_3)^2$$
$$=0.04\lambda_3^2-0.08\lambda_3-1.20$$

To minimize f we set $\frac{df}{d\lambda_3}=0$.

This gives $2\times 0.04\lambda_3-0.08=0$ or, $\lambda_3=1 \therefore \lambda_3^*=1$.

Hence $X_4=X_3+\lambda_3^* S_3=\begin{bmatrix}-0.8\\1.2\end{bmatrix}+1\begin{bmatrix}-0.2\\0.2\end{bmatrix}=\begin{bmatrix}-1.0\\1.4\end{bmatrix}$

The gradient of f at X_4 is given by

$$\nabla f_4=[\nabla f]_{X_4}=\begin{bmatrix}1+4(-1.0)+2(1.4)\\-1+2(-1.0)+2(1.4)\end{bmatrix}=\begin{bmatrix}-0.20\\-0.20\end{bmatrix}\neq\begin{bmatrix}0\\0\end{bmatrix}$$

So X_4 is also not optimum and we are to continue the iterations until we have $\nabla f_n\approx\begin{bmatrix}0\\0\end{bmatrix}$ and then X_n is taken as the optimum point.

Convergence Criteria : The following criteria can be used to terminate the iterative process.

i) $\left|\frac{f(X_{i+1})-f(X_i)}{f(X_i)}\right|\leq\epsilon$

ii) $\left|\frac{\partial f}{\partial x_i}\right|<\epsilon$ for all $i=1, 2, ..., n$

iii) $|X_{i+1}-X_i|\leq\epsilon$

The method of steepest descent may appear to be the best unconstrained minimization technique since

each one-dimensional search starts in the direction in which the function decreases at the fastest rate. But this is seen not to be true in most cases. The method is not really effective in most of the problems owing to the fact that the steepest descent direction is only a local property not global.

106.6 MODIFICATION OF THE STEEPEST DESCENT METHOD:

To speed up the convergence of the steepest descent method numerous modification have been suggested. One of these methods was suggested by Shah Buebler and Kempthorne.

In this method, the search directions are taken alternately as the steepest descent direction and the direction given by $S_i = X_i - X_{i-2}$. This method is called a gradient based PARTAN (parallel tangents) method. The algorithm of this method is stated as follows:

i) Start with an initial point X_1.

ii) Search for the minimum along the direction $S_1 = [-\nabla f]_{X_1}$ and set the new point as

$$X_2 = X_1 + \lambda_1^* S_1$$

iii) Search along the directions $S_2 = [-\nabla f]_{X_2}$ and obtain the new point $X_3 = X_2 + \lambda_2^* S_2$

iv) Find the next search direction as

$$S_3 = X_3 - X_1$$

and obtain the point $X_4 = X_3 + \lambda_3^* S_3$.

v) Take the new search direction as

$$S_i = \begin{cases} -[\nabla f]_{X_i} & \text{for } i = 4,6,8,\ldots, 2K \\ X_i - X_{i-2} & \text{for } i = 5,7,9,\ldots,2K-1 \end{cases}$$

and find the new point as

$$X_{i+1} = X_i + \lambda_i^* S_i$$

where λ_i^* is the optimal step length in the direction S_i

vi) Test whether X_{i+1} is optimum. If X_{i+1} is optimum then stop the procedure. Otherwise set new $i=i+1$ and repeat step (v) onward.

106.7 QUADRATICALLY CONVERGENT METHOD

Definition 105.7.1: A minimization method is called quadratically convergent method if it locates the minimum of general quadratic function in no more than a pre-determined number of operations and if the limitting number of operations is directly related to the number of variables.

Definition 105.7.2: Let A be an $n \times n$ symmetric matrix. A set of n vectors $S_1, S_2, \ldots, S_n$ is said to be A-conjugate directions if $S_i^T A S_j = 0$ for all $i \neq j, i, j = 1,2,3,\ldots,n$.

Example 105.7.3: Find the conjugate directions for the symmetric matrix $\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$.

Solution: Let $A = \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$ and A-conjugate directions be $S_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ and $S_2 = \begin{bmatrix} \gamma \\ \delta \end{bmatrix}$.

$\therefore S_1^T A S_2 = 0$

or, $[\alpha \ \beta]\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} \gamma \\ \delta \end{bmatrix} = 0$

or, $\gamma(2\alpha - 3\beta) + \delta(-3\alpha + 2\beta) = 0$

Let $\alpha = 1, \beta = 2, \gamma = -1$ $\quad \therefore -1(2.1 - 3.2) + \delta(-3.1 + 2.2) = 0$

or, $4 + \delta(+1) = 0$

or, $\delta = -4$

Thus the conjugate directions are $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -4 \end{bmatrix}$.

We note that for a given matrix there are many conjugate directions.

Matrix representation of a quadratic expression:

Any quadratic expression can be expressed with the help of matrices as

$$\frac{1}{2}X^T AX + B^T X + C$$

Where A is a symmetric matrix

eg. $3x_1^2+2x_2^2+4x_3^2+4x_1x_2-x_2x_3+3x_3x_1+3x_1-2x_2+x_3+7$

can be written as $\frac{1}{2}X^TAX+B^TX+C$.

Where $A=\begin{bmatrix}6 & 4 & 3\\ 4 & 4 & -1\\ 3 & -1 & 8\end{bmatrix}, B=\begin{bmatrix}3\\ -2\\ 1\end{bmatrix}, C=7, X=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$.

THEOREM 106.7.4: If a quadratic function $Q(X)=\frac{1}{2}X^TAX+B^TX+C$ is minimized sequentially once along each direction of a set of n A-conjugate directions then the global minimum of $Q(X)$ will be located at a before the nth step regardless of the starting point and the order in which the directions are used.

Proof: Let $S_1,S_2,\ldots,S_n$ be the set of A-conjugate directions.

Then we have $S_i^TAS_j=0$ for all $i\neq j$, $i,j=1,2,\ldots n$. ... (1)

We first show that $S_1,S_2,\ldots,S_n$ are linearly independent.

Let $\alpha_1S_1+\alpha_2S_2+\ldots+\alpha_nS_n=0$

Multiplying both sides from left by S_j^TA we have

$$S_j^T\left(\alpha_1S_1+\alpha_2S_2+\ldots+\alpha_jS_j+\ldots+\alpha_nS_n\right)=0$$

or, $\alpha_j\left(S_j^TAS_j\right),(by\ (1))$

or, $\alpha_j=0$

This is true for all j. $\therefore \alpha_1=0,\alpha_2=0,\ldots,\alpha_n=0$.

Hence $S_1,S_2,\ldots,S_n$ are linearly independent.

Let $Q(X)$ is minimum at $X=X^*$. Then from necessary conditions we have

$$[\nabla Q]_{X^*}=0$$

or, $AX^*+B=0$ ... (2)

Since $S_1,S_2,\ldots,S_n$ are linearly independent, they form a basis. Hence we can find constants $\beta_1,\beta_2,\ldots,\beta_n$ such that for given a point X_1 we have

$$X^*-X_1=\sum_{i=1}^{n}\beta_iS_i$$

or, $$X^* = X_1 + \sum_{i=1}^{n} \beta_i S_i \qquad \ldots (3)$$

Using (3) in (2) we get

$$AX_1 + \sum_{i=1}^{n} \beta_i AS_i + B = 0$$

Multiplying both sides from left by S_j^T we get

$$S_j^T(AX_1 + B) + \sum_{i=1}^{n} \beta_i \left(S_j^T AS_i\right) = 0$$

or, $$S_j^T(AX_1 + B) + \beta_j \left(S_j^T AS_j\right) = 0 \text{ [by (1)]}$$

or, $$\beta_j = -\frac{S_j^T(AX_1 + B)}{S_j^T AS_j} \qquad \ldots (4)$$

Now we consider an iterative minimization procedure to minimize the quadratic function $Q(X)$ starting from the point X_1 and using the search directions $S_1, S_2, \ldots, S_n$.

Then the successive points obtained are given by

$$X_{i+1} = X_i + \lambda_i^* S_i;\ i = 1,2,\ldots,n \qquad \ldots (5)$$

Where λ_i^* is found by minimizing $Q(X_i + \lambda_i S_i)$.

$\therefore$ We have $$\left[\frac{dQ(X_i + \lambda_i S_i)}{d\lambda_i}\right]_{\lambda_i^*} = 0$$

or, $$\left[\frac{dQ(y_i, y_i, \ldots y_n)}{d\lambda_i}\right]_{\lambda_i^*} = 0 \text{ where } y_i = X_i + \lambda_i S_i;\ i = 1,2,\ldots,n$$

$$= \begin{bmatrix} X_{1i} + \lambda_i S_{1i} \\ X_{2i} + \lambda_i S_{2i} \\ \ldots\ldots\ldots \\ X_{ni} + \lambda_n S_{ni} \end{bmatrix}$$

or, $$\left[\sum_{j=1}^{n} \frac{\partial Q}{\partial y_j} \frac{\partial y_j}{\partial \lambda_j}\right]_{\lambda_i^*} = 0$$

or, $$S_i^T [\nabla Q]_{\lambda_i^*} = 0$$

or, $S_i^T[AX+B]_{\lambda_i^*}=0$

or, $S_i^T\left[A\left(X_i+\lambda_i^* S_i\right)+B\right]=0$

or, $S_i^T(AX_i+B)+\lambda_i^* S_i^T AS_i=0$

or, $\lambda_i^*=-\dfrac{S_i^T(AX_i+B)}{S_i^T AS_i}$

or, $\lambda_i^*=-\dfrac{\left[S_i^T(AX_i+B)\right]^T}{S_i^T AS_i}$

or, $\lambda_i^*=-\dfrac{(AX_i+B)^T S_i}{S_i^T AS_i}$ ... (6)

From (5) we have

$$X_i=X_{i-1}+\lambda_{i-1}^* S_{i-1}$$

$$X_{i-1}=X_{i-2}+\lambda_{i-2}^* S_{i-2}$$

... ...

... ...

$$X_3=X_2+\lambda_2^* S_2$$

$$X_2=X_1+\lambda_1^* S_1$$

Adding we have

$$X_i=X_1+\sum_{j=1}^{i-1}\lambda_j^* S_j \qquad \text{... (7)}$$

Multiplying both sides from left by $X_i^T A$ we have

$$X_i^T AS_i=X_1^T AS_i+\sum_{j=1}^{i-1}\lambda_j^* S_j^T AS_i$$

Using (1) this gives

$$X_i^T AS_i=X_1^T AS_i \qquad \text{... (8)}$$

From (6) we have

$$\lambda_i^*=-\frac{(AX_i)^T S_i+B^T S_i}{S_i^T AS_i}$$

$$= -\frac{X_i^T AS_i + B^T S_i}{S_i^T AS_i} \left[\because A^T = A\right]$$

$$= -\frac{X_1^T AS_i + B^T S_i}{S_i^T AS_i} [by\ (8)]$$

$$= -\frac{(AX_1)^T S_i + B^T S_i}{S_i^T AS_i} \left[\because A^T = A\right]$$

$$= -\frac{\left[(AX_1)^T + B^T\right] S_i}{S_i^T AS_i}$$

$$= -\frac{(AX_1 + B)^T S_i}{S_i^T AS_i}$$

$= \beta_i$ [by (4)]

Thus $\beta_i = \lambda_i^*$ for all $i = 1, 2, ..., n$ (9)

$\therefore$ (3) becomes

$$X^* = X_1 + \sum_{i=1}^{n} \lambda_i^* S_i$$

By (7) this gives

$$X^* = X_{n+1}$$

This means X^* can be obtained by finding $\lambda_1^*, \lambda_2^*, ..., \lambda_n^*$ for which we need n steps or less than n steps (when some $\beta_i = 0$). Since we have not made any assumption regarding X_1 and the order of $S_1, S_2, ..., S_n$ the process is independent of the starting point as well as the order of the search directions.

THEOREM 106.7.5: Suppose that the point X_{i+1} is reached after i steps while minimizing a quadratic function $f(X) = \frac{1}{2} X^T AX + B^T X + C$. If the search directions used in the minimization process $S_1, S_2, ..., S_i$ are mutually conjugate with respect to A then $S_K^T \nabla f_{i+1} = 0$ for $k = 1, 2, ..., i$.

Proof: The gradient of the function $f(X)$ evaluated at the point X_{i+1} is given by

$$\nabla f_{i+1} = [\nabla f]_{X_{i+1}} = AX_{i+1} + B \quad ...(1)$$

If λ_j^* is the minimizing step length in the direction S_j for all j then we have

$$X_{i+1} = X_i + \lambda_i^* S_i$$

$$X_i = X_{i-1} + \lambda_{i-1}^* S_{i-1}$$

... ...

... ...

$$X_3 = X_2 + \lambda_2^* S_2$$

$$X_2 = X_1 + \lambda_1^* S_1$$

Adding we have

$$X_{i+1} = X_1 + \lambda_1^* S_1 + \lambda_2^* S_2 + \ldots\ldots + \lambda_k^* S_k + \lambda_{k+1}^* S_{k+1} + \ldots\ldots + \lambda_i^* S_i$$

From this we have

$$X_{i+1} = X_{k+1} + \sum_{j=k+1}^{i} \lambda_j^* S_j$$

or, $$AX_{i+1} + B = A\left(X_{k+1} + \sum_{j=k+1}^{i} \lambda_j^* S_j \right) + B$$

or, $$(AX_{i+1} + B) = (AX_{k+1} + B) + \sum_{j=k+1}^{i} \lambda_j^* AS_j$$

or, $$\nabla f_{i+1} = \nabla f_{k+1} + \sum_{j=k+1}^{i} \lambda_j^* AS_j \text{ [by (1)]}$$

Premultiplying by S_K^T we get

$$S_K^T \nabla f_{i+1} = S_K^T \nabla f_{k+1} + \sum_{j=k+1}^{i} \lambda_j^* S_K^T AS_j \qquad \ldots (2)$$

This is true for $k = 1, 2, \ldots, i-1$.

Now λ_k^* is the minimizing step length along the direction S_K.

$$\therefore \left[\frac{df(X_k + \lambda_k S_k)}{d\lambda_k} \right]_{\lambda_k^*} = 0$$

or, $$S_k^T [\nabla f]_{\lambda_k^*} = 0$$

or, $$S_k^T [AX + B]_{\lambda_k^*} = 0$$

or, $$S_k^T \left[A(X_k + \lambda_k^* S_k) + B \right] = 0$$

or, $$S_k^T [AX_{k+1} + B] = 0$$

or, $S_k^T \nabla f_{k+1} = 0$ ... (3)

By A-conjugate property of $S_1, S_2, \ldots, S_i$ we have

$S_k^T A S_j = 0$ for all $j = k+1, k+2, \ldots, i$. ... (4)

Using (3) and (4) we get from (2)

$S_k^T \nabla f_{i+1} = 0$ for all $k = 1, 2, \ldots, i\text{-}1$.

From (3) it is also true for $i = k$.

Hence $S_k^T \nabla f_{i+1} = 0$ for $k = 1, 2, \ldots, i$.

This proves the theorem.

106.8 CONJUGATE GRADIENT METHOD (FLETCHER-REEVES METHOD)

We have seen that any minimization method that makes use of the conjugate directions is quadratically convergent. The convergence characteristics of the steepest descent method has been greatly improved by modifying it into conjugate gradient method. This method is very useful because it ensures the minimum value of a quadratic function in n steps or less than n steps. Also we know that any general function can be approximated by a quadratic near the optimum point. So conjugate gradient method is seen to be very useful optimization method.

It is proved that the conjugate directions can be constructed by the following formula.

$$S_i = -\nabla f_i + \left(|\nabla f_i|^2 / |\nabla f_{i-1}|^2\right) S_{i-1}$$

where $\nabla f_i = [\nabla f]_{X_i}$ and $f(X) = \frac{1}{2} X^T A X + B^T X + C$.

For the minimization of general functions was first suggested by Fletcher and Reeves in the year 1964. Their algorithm can be summarized as follows:

i) Start with an arbitrary initial point X_1.

ii) Set the first search direction $S_1 = -[\nabla f]_{X_1} = -\nabla f_1$.

iii) Find the point X_2 according to the relation

$X_2 = X_1 + \lambda_1^* S_1$

where λ_1^* is the optimal step length in the direction S_1

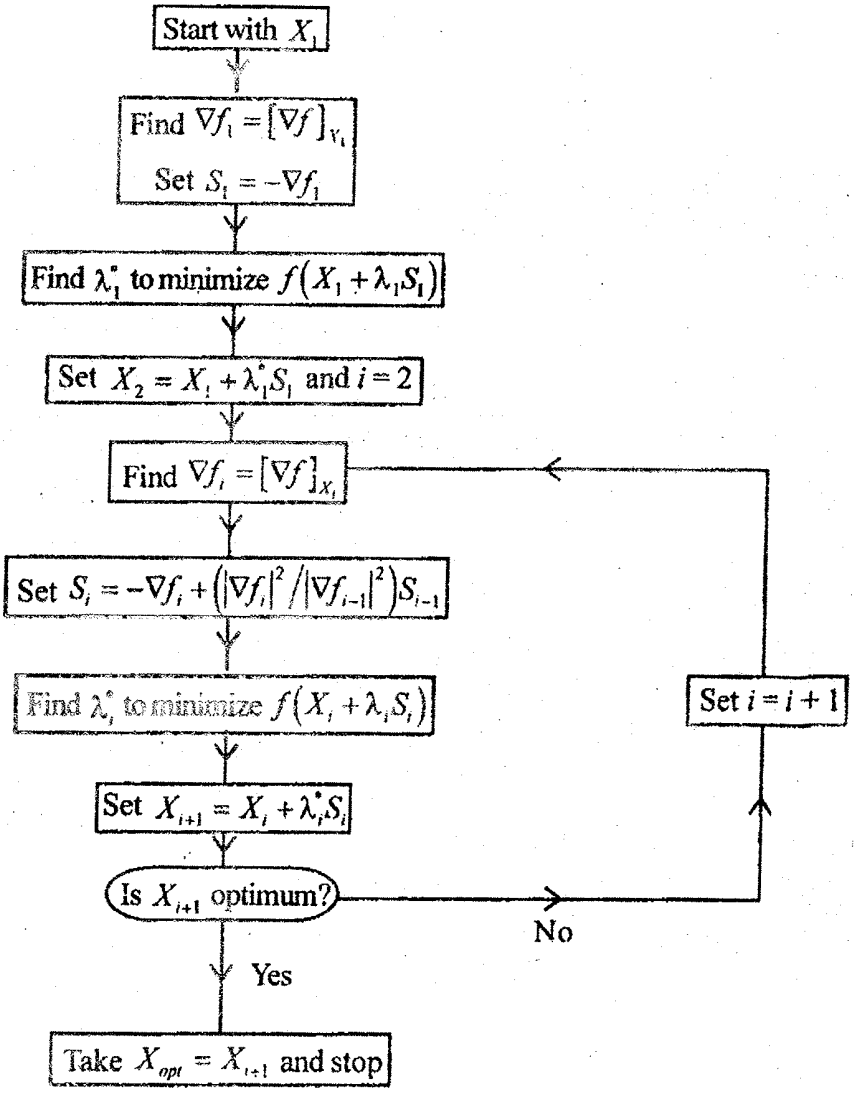
Set $i = 2$ and go to the next step.

iv) Find $\nabla f_i = [\nabla f]_{X_i}$ and set $S_i = -\nabla f_i + \left(\frac{|\nabla f_i|^2}{|\nabla f_{i-1}|^2}\right) S_{i-1}$

v) Compute the optimum step length λ_i^* in the direction S_i and find the new point $X_{i+1} = X_i + \lambda_i^* S_i$

vi) Test for the optimality of the point X_{i+1}. If X_{i+1} is optimum, stop the process. Otherwise, set the value of $i = i + 1$. Repeat steps (iv), (v) and (vi) until the convergence is achieved.

The flow chart for the Fletcher and Reeves method is shown below.



Example 106.8.1 : Using Fletcher and Reeves method

Minimize $f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$ starting from the point $X_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Solution : We have $\nabla f = \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{bmatrix} = \begin{bmatrix} 1+4x_1+2x_2 \\ -1+2x_1+2x_2 \end{bmatrix}$

$$\therefore \nabla f_1 = [\nabla f]_{X_1} = \begin{bmatrix} 1+0+0 \\ -1+0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The search direction is taken as $S_1 = -\nabla f_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

The optimal step length λ_1^* along S_1 is obtained by minimizing $f(X_1 + \lambda_1 S_1)$ with respect to λ_1.

Now $X_1 + \lambda_1 S_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\lambda_1 \\ \lambda_1 \end{bmatrix}$

$$\therefore f(X_1 + \lambda_1 S_1) = -\lambda_1 - \lambda_1 + 2(-\lambda_1)^2 + 2(-\lambda_1)(\lambda_1) + \lambda_1^2 = -2\lambda_1 + \lambda_1^2$$

We set $\frac{df}{d\lambda_1} = 0$. $\therefore \frac{d}{d\lambda_1}(-2\lambda_1 + \lambda_1^2) = 0$ or $-2 + 2\lambda_1 = 0$ or, $\lambda_1 = 1 \therefore \lambda_1^* = 1$

$$\therefore X_2 = X_1 + \lambda_1^* S_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Hence $\nabla f_2 = [\nabla f]_{X_2} = \begin{bmatrix} 1+4(-1)+2.1 \\ -1+2(-1)+2.1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

The search direction S_2 is given by

$$S_2 = -\nabla f_2 + \frac{|\nabla f_1|^2}{|\nabla f_2|^2} S_1$$

$$= -\begin{bmatrix} -1 \\ -1 \end{bmatrix} + \frac{1^2 + (-1)^2}{(-1)^2 + (-1)^2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

To find λ_2^*, we are to minimize $f(X_2 + \lambda_2 S_2)$.

Now $X_2 + \lambda_2 S_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1+2\lambda_2 \end{bmatrix}$

$$\therefore f(X_2 + \lambda_2 S_2) = (-1) - (1+2\lambda_2) + 2(-1)^2 + 2(-1)(1+2\lambda_2) + (1+2\lambda_2)^2$$

$$= -1 - 1 - 2\lambda_2 + 2 - 2 - 4\lambda_2 + 1 + 4\lambda_2 + 4\lambda_2^2$$

$$= -1 - 2\lambda_2 + 4\lambda_2^2$$

We set $\frac{df}{d\lambda_2} = 0 \;\therefore -2 + 8\lambda_2 = 0$ or, $\lambda_2 = \frac{1}{4} \;\therefore \lambda_2^* = \frac{1}{4}$.

$$\therefore X_3 = X_2 + \lambda_2^* S_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3/2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1.5 \end{bmatrix}$$

$$\text{Hence } \nabla f_3 = [\nabla f]_{X_3} = \begin{bmatrix} 1 + 4(-1) + 2 \times 1.5 \\ -1 + 2(-1) + 2 \times 1.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This shows that $X_3 = \begin{bmatrix} -1 \\ 3/2 \end{bmatrix}$ is the optimum point.

We note that the optimum point is reached in two iterations only.

106.9 NEWTON'S METHOD

If the function $f(X)$ is continuously differentiable then the local minimum point X^* is given by $[\nabla f]_{X^*} = 0$. Solving the set of n nonlinear equations $\nabla f = 0$ we get the optimal point X^*.

Newton's method: To get the minimum point X^* of the continuously differentiable function $f(X)$ we are to solve the n nonlinear equation $\nabla f = 0$. To solve these n-nonlinear equations by the Newton's method, we first linearize the set of equations about the ith approximation X_i to the minimum point X^* of f.

Let $X^* = X_i + S$ and $\nabla f = g$

From $[\nabla f]_{X^*} = 0$ we have $g(X^*) = 0$ or $g(X_i + S) = 0$

By Taylor's series expansion we get

$g(X_i) + [J]_{X_i} S + \ldots = 0$ where $[J]_{X_i}$ is the matrix of second partial derivatives of f evaluated at the point X_i.

Neglecting the higher order terms we get

$g(X_i) + [J]_{X_i} S = 0$

or, $g_i + J_i S = 0$ where $g(X_i) = g_i$ and $[J]_{X_i} = J_i$.

If J_i is non singular, then we have

$$S = -J_i^{-1} g_i$$

But the higher order terms are not negligible in general. Hence an iterative procedure has to be used to find

the improved approximations. The iterative scheme is given by

$$X_{i+1} = X_i + S_i = X_i - J_i^{-1} g_i$$

If J_1 is nonsingular then it can be shown that the sequence of points $X_1, X_2, \ldots X_i, \ldots$ converges to the actual solution X^* from any initial point X_1 sufficiently close to the solution X^*.

Theorem 106.9.1: If $f(X)$ is a quadratic then the minimum point can be obtained in a single step by Newton's method.

Proof.: Let $f(X) = \frac{1}{2} X^T AX + B^T X + C$ and the minimum point be X^*. Then $[\nabla f]_{X^*} = 0$ or, $[AX + B]_{X^*} = 0$ or $AX^* + B = 0$ or $X^* = -A^{-1}B$.

From $f(X) = \frac{1}{2} X^T AX + B^T X + C$ we have

$\nabla f = AX + B$ and J = matrix of second partial derivatives of $f = A$. By Newton's method we have

$$\begin{aligned} X_{i+1} &= X_i - J_i^{-1} g_i \\ &= X_i - A^{-1}(AX_i + B) \\ &= X_i - A^{-1}AX_i - A^{-1}B \\ &= X_i - X_i - A^{-1}B \\ &= -A^{-1}B = X^* \end{aligned}$$

$\therefore X_2 = -A^{-1}B = X^*$ for any starting point X_1.

Thus the answer is obtained in a single step.

Example 106.9.2: Using Newton's method

Minimize $f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$ with $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as starting point.

Solution: Here $f = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$

$$\therefore \frac{\partial f}{\partial x_1} = 1 + 4x_1 + 2x_2, \quad \frac{\partial f}{\partial x_2} = -1 + 2x_1 + 2x_2$$

$$\frac{\partial^2 f}{\partial x_1^2} = 4, \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = 2, \quad \frac{\partial^2 f}{\partial x_2^2} = 2$$

The starting point is $X_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$$\nabla f_1 = [\nabla f]_{X_1} = \begin{bmatrix} 1+0+0 \\ -1+0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

& $J_1 = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$

$$\therefore J_1^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$$

We have $X_2 = X_1 - J_1^{-1}\nabla f_1$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\begin{bmatrix} \frac{1}{2} & +\frac{1}{2} \\ -\frac{1}{2} & -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3/2 \end{bmatrix}$$

Now $\nabla f_2 = [\nabla f]_{X_2} = \begin{bmatrix} 1+4(-1)+2(3/2) \\ -1+2(-1)+2(3/2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

As $\nabla f_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} -1 \\ 3/2 \end{bmatrix}$ is the optimum point.

When $f(X)$ is not quadratic function then the Newton's method may not converge always. Also it may converge to a saddle point. However, the method can be improved by replacing

$$X_{i+1} = X_i - J_i^{-1}g_i, \text{ by } X_{i+1} = X_i - \lambda_i^* J_i^{-1}g_i$$

where λ_i^* is the minimizing step length in the direction $S_i = -J_i^{-1}g_i$.

This modification has following advantages.

i) It will find the minimum point in lesser number of steps compared to the original method.

ii) It finds the minimum point in all the cases whereas the original method may not converge in some cases.

iii) It usually avoids convergence to a saddle point or a maximum.

Inspite of all these advantages, this method is not generally used in practice due to the following reasons.

i) It requires the storing of the $n \times n$ matrix J_i.

ii) It becomes very difficult and sometimes even impossible to compute the elements of the matrix J_i.

iii) It requires the inversion of the matrix J_i at each step which is often difficult.

iv) It requires the evaluation of the quantity $J_i^{-1}g_i$ at each step which involves complicated calculations.

106.10 QUASI-NEWTON METHODS

We know that the differences in the function value contain information about the first derivatives. Similarly, the differences in the gradient value contain information about the second derivatives. This idea will help us to calculate approximate values of the first and second derivatives of the function. The calculations of the components of J_i in the Newton's method is thus reduced.

Let the minimum point of $f(X)$ be X^* & X_i be approximate value of X^*. Let $X^* = X_i + S$.

As X^* is minimum point we have $[\nabla f]_{X^*} = 0$ i.e. $g(X^*) = 0$

where $g(X) = \nabla f(X)$.

Thus $g(X_i + S) = 0$. Using Taylor's theorem and neglecting higher order terms we get

$$g(X^*) = g(X_i) + [J]_{X_i} S \simeq 0$$

Using $X^* = X_i + S$ we get

$$g(X^*) = g(X_i) + [J]_{X_i}(X^* - X_i) \simeq 0$$

Replacing X^* by the new approximation X_{i+1} we get

$$g(X_{i+1}) = g(X_i) + [J]_{X_i}(X_{i+1} - X_i) \simeq 0$$

or, $g_{i+1} = g_i + J_i(X_{i+1} - X_i) \simeq 0$ where $g_i = g(X_i)$ and $J_i = [J]_{X_i}$

Thus the relation between X_i and X_{i+1} is

$$g_{i+1} - g_i = J_i(X_{i+1} - X_i)$$

or, $G_i = J_i S_i$

where $G_i = g_{i+1} - g_i$ and $S_i = X_{i+1} - X_i$

If the matrix J_i is non-singular then we have

$$S_i = J_i^{-1} - G_i$$

This equation permits us to use the gradient difference G_i to build up an approximation to J_i^{-1}.

To develop the iterative scheme of quasi-Newton methods, let

$$S_i = H_i G_i \qquad (1)$$

where H_i is the approximation to J_i^{-1} in the ith step.

Let the point X_{K+1} found at the end of kth step be the stationary point. Then we have $g_{K+1} = [\nabla f]_{X_{K+1}} = 0$

From (1) we have

$$S_K = H_K G_K = H_K(g_{K+1} - g_K) = -H_K g_K [\because g_{K+1} = 0]$$

But, in general, X_{K+1} will be a stationary point, it will be an approximation to the stationary point. Hence we use $S_K = -H_K g_K$ as a direction of search and find the new point X_{K+1} as

$$X_{K+1} = X_K + S_K \tag{2}$$

where $S_K = -\lambda_K^* H_K g_K$ (3)

Here λ_K^* is the minimizing step length along the direction $-H_K g_K$.

All the quasi-Newton methods are based on the equations (2) and (3). They differ from one another only in the methods they use in constructing H_K and in finding λ_K^*. It is very important point to note that the method of constructing H_K completely eliminates the need for evaluating second derivatives and performing matrix inversions. It is found that the sequence $\{X_K\}$, in general, converges to the minimum point X^*. Also it is seen that the matrix H_K is improved at each iteration to H_{K+1} and the sequence $\{H_K\}$ converges to J^{-1}.

106.11 DAVIDON-FLETCHER-POWELL METHOD (VARIABLE METRIC METHOD)

Davidon-Fletcher-Powell method is an important quasi-Newton method. This method is the best general purpose unconstrained optimization technique making use of the derivatives.

The iterative procedure of this method is as follows:

i) Start with an initial point X_1 and a $n \times n$ positive definite symmetric matrix H_1. Usually H_1 is taken as the identity matrix I. Set iteration number as $i = 1$.

ii) Compute the gradient of the function f at the point X_1 i.e. compute $\nabla f_1 = [\nabla f]_{X_1}$.

Take $S_i = -H_i \nabla f_i$ as the search direction at X_i.

iii) Find the optimal step length λ_i^* in the direction S_i and set $X_{i+1} = X_i + \lambda_i^* S_i$.

iv) Test the new point X_{i+1} for optimality. If X_{i+1} is optimal, terminate the iterative process. Otherwise go to step (v).

v) Update H_i to H_{i+1} as

$$H_{i+1} = H_i + M_i + N_i$$

where $M_i = \left(\lambda_i^* S_i S_i^T\right) / \left(S_i^T Q_i\right)$

$$N_i = -\left(H_i Q_i\right)\left(H_i Q_i\right)^T / \left(Q_i^T H_i Q_i\right)$$

$$Q_i = \nabla f_{i+1} - \nabla f_i$$

vi) Set the new iteration number $i = i + 1$ and go to step (ii).

106.12 ILLUSTRATIVE EXAMPLES

Example 105.12.1 : Using Davidon-Fletcher-Powell method

minimize $f(x_1, x_2) = 2x_1^2 + 4x_2^2 - 12x_1 + 16x_2 + 41$

with $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as starting point.

Solution: Here $f = 2x_1^2 + 4x_2^2 - 12x_1 + 16x_2 + 41$

$$\therefore \nabla f = \begin{bmatrix} \partial f / \partial x_1 \\ \partial f / \partial x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 - 12 \\ 8x_2 + 16 \end{bmatrix}$$

Thus $\nabla f_1 = [\nabla f]_{x_1} = \begin{bmatrix} 4 - 12 \\ 8 + 16 \end{bmatrix} = \begin{bmatrix} -8 \\ 24 \end{bmatrix}$

We take $H_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore S_1 = -H_1 \nabla f_1 = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -8 \\ 24 \end{bmatrix} = \begin{bmatrix} -8 \\ 24 \end{bmatrix}$$

To find the minimizing step length λ_1^* along S_1, we minimize

$$f(X_1 + \lambda_1 S_1) = f(1 + 8\lambda_1, 1 - 24\lambda_1)$$

$$= 2(1 + 8\lambda_1)^2 + 4(1 - 24\lambda_1)^2 - 12(1 + 8\lambda_1) + 16(1 - 24\lambda_1) + 41$$

$$= 2 + 32\lambda_1 + 128\lambda_1^2 + 4 - 192\lambda_1 + 2304\lambda_1^2 - 12 - 96\lambda_1 + 16 - 384\lambda_1 + 41$$

$$= 2432\lambda_1^2 - 640\lambda_1 + 51$$

We set $\frac{df}{d\lambda_1} = 0$

$$\therefore 2432 \times 2\lambda_1 - 640 = 0 \text{ or, } \lambda_1 = \frac{640}{2 \times 2432} = \frac{10}{76} = 0.1316$$

$$\therefore \lambda_1^* = 0.1316$$

$\therefore$ The second approximation is given by

$$X_2 = X_1 + \lambda_1^* S_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0.1316 \begin{bmatrix} 8 \\ -24 \end{bmatrix} = \begin{bmatrix} 2.0528 \\ -2.1584 \end{bmatrix}$$

Now $\nabla f_2 = [\nabla f]_{X_2} = \begin{bmatrix} 4 \times 2.0528 - 12 \\ 8 \times (-2.1584) + 16 \end{bmatrix} = \begin{bmatrix} -3.7888 \\ -1.2672 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore X_2$ is not optimum point.

To update the matrix H_1 we compute

$$Q_1 = \nabla f_2 - \nabla f_1 = \begin{bmatrix} -3.7888 \\ -1.2672 \end{bmatrix} - \begin{bmatrix} -8 \\ 24 \end{bmatrix} = \begin{bmatrix} 4.2112 \\ -25.2672 \end{bmatrix}$$

$$\therefore S_1^T Q_1 = \begin{bmatrix} 8 & -24 \end{bmatrix} \begin{bmatrix} 4.2112 \\ -25.2672 \end{bmatrix} = 640.1024$$

$$S_1 S_1^T = \begin{bmatrix} 8 \\ -24 \end{bmatrix} \begin{bmatrix} 8 & -24 \end{bmatrix} = \begin{bmatrix} 64 & -192 \\ -192 & 576 \end{bmatrix}$$

$$H_1 Q_1 = Q_1 = \begin{bmatrix} 4.2112 \\ -25.2672 \end{bmatrix}$$

$$\therefore (H_1 Q_1)(H_1 Q_1)^T = \begin{bmatrix} 4.2112 \\ -25.2672 \end{bmatrix} \begin{bmatrix} 4.2112 & -25.2672 \end{bmatrix}$$

$$= \begin{bmatrix} 17.7242 & -106.4052 \\ -106.4052 & 638.4314 \end{bmatrix}$$

$$\text{Also } Q_1^T (H_1 Q_1) = \begin{bmatrix} 4.2112 & -25.2672 \end{bmatrix} \begin{bmatrix} 4.2112 \\ -25.2672 \end{bmatrix} = 656.1656$$

$$\therefore N_1 = -\frac{(H_1 Q_1)(H_1 Q_1)^T}{Q_1^T (H_1 Q_1)} = -\frac{1}{656.1656} \begin{bmatrix} 17.7242 & -106.4052 \\ -106.4052 & 638.4314 \end{bmatrix}$$

$$= -\begin{bmatrix} 0.027 & -0.1625 \\ -0.1625 & 0.973 \end{bmatrix}$$

$$M_1 = \frac{\lambda_1^* S_1 S_1^T}{S_1^T Q_1} = \frac{0.1316}{640.1024}\begin{bmatrix} 64 & -192 \\ -192 & 576 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0132 & -0.0395 \\ -0.0395 & 0.1184 \end{bmatrix}$$

$$\therefore H_2 = H_1 + M_1 + N_1$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0.0132 & -0.0395 \\ -0.0395 & 0.1184 \end{bmatrix} + \begin{bmatrix} -0.027 & 0.1625 \\ 0.1625 & -0.973 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8062 & 0.123 \\ 0.123 & 0.1454 \end{bmatrix}$$

Hence $S_2 = -H_2 \nabla f_2 = -\begin{bmatrix} 0.8062 & 0.123 \\ 0.123 & 0.1454 \end{bmatrix}\begin{bmatrix} -3.7888 \\ -1.2672 \end{bmatrix} = \begin{bmatrix} 3.21 \\ 0.622 \end{bmatrix}$

To find the minimizing step length along S_2 we are to minimize

$$f(X_2 + \lambda_2 S_2) = f(3.21\lambda_2 - 0.9472, 0.622\lambda_2 - 0.1584)$$

$$= 2(3.21\lambda_2 - 0.9472)^2 + 4(0.622\lambda_2 - 0.1584) - 12(3.21\lambda_2 - 0.9472)$$

$$+16(0.622\lambda_2 - 0.1584) + 41$$

We set $\frac{df}{d\lambda_2} = 0$.

This gives $\lambda_2 = 0.292$

$$\therefore \lambda_2^* = 0.292$$

The third approximation is given by

$$X_3 = X_2 + \lambda_2^* S_2 = \begin{bmatrix} 2.0528 \\ -2.1584 \end{bmatrix} + 0.292\begin{bmatrix} 3.21 \\ 0.622 \end{bmatrix}$$

$$= \begin{bmatrix} 2.99 \\ -1.98 \end{bmatrix}$$

Now $\nabla f_3 = [\nabla f]_{X_3} = \begin{bmatrix} 4 \times 2.99 - 12 \\ 8 \times (-1.98) + 16 \end{bmatrix} = \begin{bmatrix} -0.04 \\ 0.16 \end{bmatrix} \simeq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\therefore X_3 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2.99 \\ -1.98 \end{bmatrix} \text{ i.e. } x_1 = 2.99,\ x_2 = -1.98$$

is the optimum point.

106.13 CONSTRAINED OPTIMIZATION TECHNIQUES

The constrained optimization problem is

Minimize $f(X)$

subject to $g_j(X) \le 0,\ j = 1,2,\ldots,m$

There are many techniques to solve a constrained non linear programming problem. All these methods can be classified as follows

Constrained optimization techniques

Direct methods	Indirect methods
i) Heuristic search methods	i) By the transformation of variables
ii) Methods of feasible directions	ii) Penalty function methods
a) Zoutendijlis method	a) Interior penalty function methods
b) Gradient projection method	b) Exterior penalty function methods
iii) Cutting plane method	

In the direct method, the constraints are handled in an explicit manner whereas in most of the indirect methods, the constrained problem is solved as a sequence of unconstrained minimization problems.

106.14 METHODS OF FEASIBLE DIRECTIONS

Definition 106.14.1: A direction is said to be a feasible direction if at least a small step can be taken without leaving the feasible domain.

Definition 106.14.2: A feasible direction is said to be a usable feasible direction if the objective function value can be reduced at least by a small amount along this.

Methods of Feasible Directions:

The basic idea of the methods of feasible direction is to choose a starting point satisfying all the constraints and to move to a point according to the iterative scheme

$$X_{i+1} = X_i + \lambda_i^* S_i$$

Where X_i is the starting point in the feasible region for the ith iteration

S_i is the direction of movement

λ_i^* is the distance of movement

X_{i+1} is the final point in the feasible region obtained at the end of ith iteration.

The value of λ_i^* is to be chosen so that X_{i+1} remains in the feasible region. The search direction S_i is to be found such that (i) it is feasible direction i.e. a small move does not violates any constraint, and (ii) it is usable feasible direction i.e. the value of the objective function reduces in that direction at least by a small amount. The new point X_{i+1} is taken as the starting point for the next iteration and the whole procedure is repeated several times until a point is obtained such that no direction satisfying both (i) and (ii) can be found.

There are many ways of choosing usuable feasible directions and hence there are many different methods of feasible directions.

We note that the objective function $f(X)$ is to minimize an constraints are $g_j(X) \le 0$, $j = 1,2,...,m$. Hence a direction vector S_i will be a usable feasible direction if it satisfies the relations

$$\left[\frac{d}{d\lambda} f(X_i + \lambda S_i)\right]_{\lambda=0} \le 0 \text{ i.e. } S_i^T [\nabla f]_{X_i} \le 0$$

and $$\left[\frac{d}{d\lambda} g_j(X_i + \lambda S_i)\right]_{\lambda=0} \le 0 \text{ i.e. } S_i^T [\nabla g_j]_{X_i} \le 0 \text{ for all } j = 1, 2, ..., m.$$

106.15 ZOUTENDIJK'S METHOD

The methods of feasible directions differ from one another only in the manner in which they generate usable feasible directions. In Zoutendijk's method, the usable feasible direction is taken as the negative of the gradient direction if the initial point of the iteration lies in the interior (not on the boundary) of the feasible region. However, if the initial point lies on the boundary of the feasible region, some constraints will be active and the usable feasible direction is found so as to satisfy $S_i^T [\nabla f]_{X_i} \le 0$ and $S_i^T [\nabla g_j]_{X_i} \le 0$ for all $j = 1, 2, ..., m$.

The iterative procedure of Zoutendijk's method of feasible directions is stated as follows:

106.15.1 Algorithm of Zoutendijk's method:

i) Start with an initial feasible point X_1 and small numbers ϵ_1, ϵ_2 and ϵ_3 to test the convergence of the method. Evaluate $f(X_1)$ and $g_j(X_1)$, $j = 1,2,...,m$. Set the iteration number as $i = 1$.

ii) If $g_j(X_i) < 0$ for all $j = 1,2,...,m$ i.e. if X_i is an interior feasible point then set the current search direction as

$$S_i = -[\nabla f]_{X_i}$$

Normalize S_i in a suitable manner, and go to step (V). If at least one $g_j(X_i) = 0$, go to step (iii).

iii) Find a usable feasible direction S by solving the direction finding problem:

Minimize $-\alpha$

Subject to $$S^T[\nabla g_j]_{X_i} + \theta_j \alpha \le 0, \quad j = 1,2,...,p$$

$$S^T[\nabla f]_{X_i} + \alpha \le 0$$

$$-1 \le s_i \le 1, \quad k = 1,2,...,n$$

where s_i is the ith component of S, the first p constraints have been assumed to be active at the point X_i, and the values of all θ_j can be taken as unity. Here α can be taken as an additional design variable.

iv) If the value of α^* found in step (iii) is very nearly equal to zero, that is, if $\alpha^* \le \epsilon_1$, terminate the computation by taking $X_{opt} \simeq X_i$. If $\alpha^* > \epsilon_1$, go to step (v) by taking $S_i = S$.

v) Find a suitable step length λ_i^* along the direction S_i and obtain a new point X_{i+1} as

$$X_{i+1} = X_i + \lambda_i^* S_i$$

vi) Evaluate the objective function $f(X_{i+1})$

vii) Test for the convergence as

If $\left|\dfrac{f(X_i) - f(X_{i+1})}{f(X_i)}\right| \le \epsilon_2$ and $|X_i - X_{i+1}| \le \epsilon_3$ terminate the iteration by taking $X_{opt} = X_{i+1}$.

Otherwise, go to step (viii).

viii) Set the new iteration number as $i = i + 1$, and repeat from step (ii) onwards.

106.16 ROSEN'S GRADIENT PROJECTION METHOD

The gradient projection method of Rosen is an alternative to the Zoutendijk's method. It does not require the solution of an auxiliary linear optimization problem to find the usable feasible direction. It uses the projection of the negative of the objective function gradient onto the constraints that are currently active.

The iterative procedure of Rosen's Gradient Projection method is stated as follows.

106.16.1 Algorithm of Rosen's Gradient Projection method

i) Start with an initial point X_1. The point X_1 has to be feasible, i.e. $g_j(X_i) \le 0,\ j = 1,2,\ldots,m.$

ii) Set the iteration number as $i = 1$.

iii) If X_i is an interior feasible point i.e., if $g_j(X_i) < 0$ for $j = 1, 2, \ldots, m$, set the direction of search as $S_i = -[\nabla f]_{X_i} = -\nabla f(X_i)$ normalize the search direction as $S_i = -\nabla f(X_i)/\|\nabla f(X_i)\|$ and go to step (v).

However, if $g_j(X_i) = 0$ for $j = j_1, j_2, \ldots, j_p$, go to step (iv).

iv) Calculate the projection matrix P_i as

$$P_i = I - N_p\left(N_p^T N_p\right)^{-1} N_p^T$$

where $N_p = -\left[\nabla g_{j_1}(X_i) \nabla g_{j_2}(X_i) \ldots \nabla g_{j_p}(X_i)\right]$

and find the normalized search direction S_i as

$$S_i = -P_i \nabla f(X_i)/\|P_i \nabla f(X_i)\|$$

v) Test whether $S_i = 0$ or not. If $S_i \neq 0$, go to step (vi).

If $S_i = 0$, compute the vector $\vec{\lambda}$ at X_i as

$$\vec{\lambda} = -\left(N_p^T N_p\right)^{-1} N_p^T \nabla f(X_i)$$

If all the components of the vector $\vec{\lambda}$ are non negative, take $X_{opt} = X_i$ and stop the iterative procedure.

If some of the components of $\vec{\lambda}$ are negative, find the component λ_q that has the most negative value and form the new matrix N_p as

$N_p = \left[\nabla g_{j_1}\ \nabla g_{j_2} \ldots \nabla g_{j_{q-1}}\ \nabla g_{j_{q+1}} \ldots \nabla g_{j_p}\right]$ and go to step (iii)

vi) If $S_i \neq 0$, find the maximum step length λ_M that is permissible without violating any of the constraints as

$\lambda_M = \min(\lambda_K), \lambda_K > 0$ and k is any integer among 1 to m other than $j_1, j_2, \ldots, j_p$.

Also find the value of $\left[\frac{df}{d\lambda}\right]_{\lambda_M} = S_i^T \nabla f(X_i + \lambda_M S_i)$.

If $\left[\frac{df}{d\lambda}\right]_{\lambda_M}$ is zero or negative, take the step length as $\lambda_i = \lambda_M$. On the other hand if $\left[\frac{df}{d\lambda}\right]_{\lambda_M}$ is positive, find the minimizing step length λ_i^* either by interpolation or by any of the methods, and take $\lambda_i = \lambda_i^*$.

vii) Find the new approximation to the minimum as

$X_{i+1} = X_i + \lambda_i S_i$

If $\lambda_i = \lambda_M$ or if $\lambda_M \leq \lambda_i^*$ some new constraints become active at X_{i+1} and hence generate the new matrix N_p to include the gradients of all active constraints evaluated at X_{i+1}. Set the new iteration number as $i = i + 1$, and go to step (iv).

If $\lambda_i = \lambda_i^*$ and $\lambda_i^* < \lambda_M$, no new constraint will be active at X_{i+1} and hence the matrix N_p remains unaltered. Set the new value of i as $i = i + 1$ and go to step (iii).

106.17 ILLUSTRATIVE EXAMPLES

Example 106.17.1: Using Rosen's Gradient Projection method

Minimize $f = x_1^2 + x_2^2 - 2x_1 - 4x_2$

subject to $x_1 + 4x_2 \leq 5$

$2x_1 + 3x_2 \leq 6$

$x_1, x_2 \geq 0$

starting from the point (1,1).

Solution: The problem is

Minimize $f = x_1^2 + x_2^2 - 2x_1 - 4x_2$

subject to $g_1 = x_1 + 4x_2 - 5 \leq 0$

$$g_2 = 2x_1 + 3x_2 - 6 \le 0$$

$$g_3 = -x_1 \le 0$$

$$g_4 = -x_2 \le 0$$

The starting point is $X_1 = \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix}$ and $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 - 2 \\ 2x_2 - 4 \end{bmatrix}$

We have $g_1(X_1) = 1 + 4 - 5 = 0,\ g_2(X_1) = 2 + 3 - 6 = -1 < 0$

$g_3(X_1) = -1 < 0,\ g_4(X_1) = -1 < 0$

Since $g_1(X_1) = 0$ we have $p = 1$ and $j_1 = 1$

Now $N_1 = \nabla g_1(X_1) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

$\therefore$ The projection matrix is given by

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \end{bmatrix} \left[\begin{bmatrix} 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right]^{-1} \begin{bmatrix} 1 & 4 \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 16 & -4 \\ -4 & 1 \end{bmatrix}$$

The search directoin S_1 is obtained as follows

$$-P_1[\nabla f]_{X_1} = -\frac{1}{17} \begin{bmatrix} 16 & -4 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 2 \times 1 - 2 \\ 2 \times 1 - 4 \end{bmatrix} = \begin{bmatrix} -8/17 \\ 2/17 \end{bmatrix} = \begin{bmatrix} -0.4707 \\ 0.1177 \end{bmatrix}$$

The normalized search direction S_1 is given by

$$S_1 = \frac{-P_1 \nabla f(X_1)}{\| P_1 \nabla f(X_1) \|}$$

$$= \frac{1}{\left[(-0.4707)^2 + (0.1177)^2 \right]^{1/2}} \begin{bmatrix} -0.4707 \\ 0.1177 \end{bmatrix} = \begin{bmatrix} -0.9701 \\ 0.2425 \end{bmatrix}$$

Since $S_1 \neq 0$, to find the step length λ_M, we set

$$X = X_1 + \lambda S_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} -0.9701 \\ 0.2425 \end{bmatrix} = \begin{bmatrix} 1 - 0.9701\lambda \\ 1 + 0.2425\lambda \end{bmatrix}$$

For $j = 2$ we have

$$g_2(X) = (2.0 - 1.9402\lambda) + (3.0 + 0.7275\lambda) - 6.0 = 0 \text{ at}$$

$$\lambda = \lambda_2 = -0.8245$$

For $j = 3$ we have

$$g_3(X) = -(1.0 - 0.9701\lambda) = 0 \text{ at } \lambda = \lambda_3 = 1.03$$

For $j = 4$ we have

$$g_4(X) = -(1.0 + 0.2425\lambda) = 0 \text{ at } \lambda = \lambda_4 = -4.124$$

$$\therefore \lambda_M = \lambda_3 = 1.03$$

Also $f(X) = (1.0 - 0.9701\lambda)^2 + (1.0 + 0.2425\lambda)^2$

$$-2(1.0 - 0.9701\lambda) - 4(1.0 + 0.2425\lambda)$$

$$= 0.9998\lambda^2 - 0.4850\lambda - 4.0$$

$$\therefore \frac{df}{d\lambda} = 1.9996\lambda - 0.4850$$

$$\therefore \left[\frac{df}{d\lambda}\right]_{\lambda_M} = 1.9996 \times 1.03 - 0.4850 = 1.5746 > 0$$

As $\left[\frac{df}{d\lambda}\right]_{\lambda_M} > 0$, we compute the minimizing step length λ_1^* by setting $\frac{df}{d\lambda} = 0$.

This gives $\lambda_1 = \frac{0.4850}{1.9996} = 0.2425$

We thus obtain the new point X_2 as

$$X_2 = X_1 + \lambda_1 S_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0.2425 \begin{bmatrix} -0.9701 \\ 0.2425 \end{bmatrix} = \begin{bmatrix} 0.7647 \\ 1.0588 \end{bmatrix}$$

Since $\lambda_1 = \lambda_1^*$ and $\lambda_1^* < \lambda_M$, no new constraint has become active at X_2 and hence the matrix N_1 remains unaltered.

Since $g_1(X_2) = 0$, we set $p = 1, j_1 = 1$.

Thus we have $N_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$$P_2 = \frac{1}{17}\begin{bmatrix} 16 & -4 \\ -4 & 1 \end{bmatrix}$$

$$\nabla f(X_2) = \begin{bmatrix} 1.5294 & -2.0 \\ 2.1176 & -4.0 \end{bmatrix} = \begin{bmatrix} -0.4706 \\ -1.8824 \end{bmatrix}$$

$$\therefore S_2 = -P_2 \nabla f(X_2) = \frac{1}{17}\begin{bmatrix} 16 & -4 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} -0.4706 \\ -1.8824 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix}$$

Since $S_2 = 0$, we compute $\vec{\lambda}$ at X_2 as

$$\vec{\lambda} = -\left(N_1^T N_1\right)^{-1} N_1^T \nabla f(X_2) = -\frac{1}{17}[1\ 4]\begin{bmatrix} -0.4706 \\ -1.8824 \end{bmatrix} = 0.4707 > 0$$

The non-negative component of λ indicates that we have reached the optimum point and hence

$$X_{opt} = X_2 = \begin{bmatrix} 0.7647 \\ 1.0588 \end{bmatrix}$$

and $f_{opt} = -4.059$

106.18 CUTTING PLANE METHOD

In the cutting plane method, the nonlinear constraints are linearized by using Taylor's series expansion thereby approximating the feasible region by linearized envelopes. Assuming that the objective function is linear, we can solve the approximating *LPP* by the simplex method. If the solution of the *LPP* is not sufficiently accurate, we relinearize the binding constraints about the current point and formulate a new approximating *LPP* and solve it using the simplex method. We repeat this procedure until a sufficiently accurate solution is found. We note that the approximating linear constraint cut off a portion of the existing feasible region. Hence the method is called cutting plane method.

To apply cutting plane method it is necessary that the objective function is linear. If the objective function is non-linear then we can formulate an equivalent optimization problem with linear objective function as follows.

Let the given problem be

Find $(x_1, x_2, \ldots, x_n)$ which minimize $f(x_1, x_2, \ldots, x_n)$

subject to the constraints $g_j(x_1, x_2, \ldots, x_n) \le 0,\ j = 1,2,\ldots,m.$

We introduce a new variable x_{n+1} and transform this problem into an equivalent problem as follows

Find $(x_1, x_2, \ldots, x_n, x_{n+1})$ which minimize $ox_1 + ox_2 + \ldots + ox_n + x_{n+1}$

subject to the constraints $g_j(x_1, x_2, \ldots, x_n) \leq 0 \quad j = 1,2,\ldots,m$

and $\quad g_{m+1}(x_1, x_2, \ldots, x_{n+1}) = f(x_1, x_2, \ldots, x_n) - x_{n+1} \leq 0$

Thus, without loss of generality, we can assume that the given problem is

Minimize $f(X) = f(x_1, x_2, \ldots, x_n) = C^T X = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

subject to the constraints $g_j(X) = g_j(x_1, x_2, \ldots, x_n) \leq 0, \; j = 1,2,\ldots,m$

The iterative procedure of cutting plane method can be stated as follows:

106.18.1 Algorithm of Cutting Plane method:

i) Start with an initial point X_1 and set the iteration number as $i = 1$. The point X_1 need not be feasible.

ii) Linearize the non linear constraint functions $g_j(X)$ about the point X_i as

$$g_j(X) \simeq g_j(X_i) + \left[\nabla g_j(X_i)\right]^T (X - X_i), \; j = 1,2,\ldots,m$$

iii) Formulate the approximating linear programming problem as

Minimize $f(X) = C^T X$

subject to $g_j(X_i) + \left[\nabla g_j(X_i)\right]^T (X - X_i) \leq 0, \; j = 1,2,\ldots,m$

iv) Solve the approximating *LPP* to obtain the solution vector X_{i+1}.

v) Evaluate the original constraints at X_{i+1} i.e. find $g_j(X_{i+1})$ for all $j = 1,2,\ldots,m$

vi) If $g_j(X_{i+1}) \leq \epsilon$ for all $j = 1,2,\ldots,m$ where ϵ is a prescribed small positive tolerance then all the original constraints can be assumed to have been satisfied. Hence stop the procedure and take

$X_{opt} = X_{i+1}$

If $g_{jl}(X_{i+1}) > \epsilon$ for some value of j, find the most violated constraint as

$$g_K(X_{i+1}) = \max_j \left[g_j(X_{i+1})\right]$$

Relinearize the constraint $g_k(X) \leq 0$ about the point X_{i+1} as

$$g_k(X) \simeq g_K(X_{i+1}) + \left[\nabla g_K(X_{i+1})\right]^T (X - X_{i+1}) \leq 0$$

and add this linear constraint to the previous approximating *LPP*.

vii) Set the new iteration number $i = i+1$ and increase the total number of constraints in the new approximating *LPP* by one and go to step (iv).

Note : To avoid the unbounded solution of the first approximating *LPP* we may take the first approximating *LPP* as

Minimize $f(X) = C^T X$

subject to $l_i \le x_i \le u_i,\ i = 1,2,...,n$

where l_i and u_i are chosen as lower and upper bounds of x_i and take the optimum solution of this first approximating *LPP* as X_1 in the first step.

106.19 ILLUSTRATIVE EXAMPLES

Example 106.19.1: Using cutting plane method

Maximize $f(x_1, x_2) = 7 - 2x_1 - 4x_2$

subject to $(x_1 - 4)^2 + 2(x_2 - 3)^2 \le 12$ taking $\epsilon = 0.03$

$x_1 + 2x_2 \le 6$

$1 \le x_1 \le 6$

$1 \le x_2 \le 6$

Solution : We first consider the *LPP*

Maximize $f(x_1, x_2) = 7 - 2x_1 - 4x_2$

subject to $x_1 + 2x_2 \le 6$

$1 \le x_1 \le 6$

$1 \le x_2 \le 6$

The extreme point of the feasible region are $A(1,1)$, $B(4,1)$ and $C(1,5/2)$.

The value of the objective functions are

$$f(1,1) = 1, f(4,1) = -5, f(1,5/2) = -5$$

$\therefore$ The optimal solution of this *LPP* is (1, 1)

$\therefore$ The first approximating point is $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Let $g(x_1,x_2)=(x_1-4)^2+2(x_2-3)^2-12$

$\therefore$ The given non-linear constraint is $g(x_1,x_2)\le 0$.

We have $\nabla g(X)=\begin{bmatrix}2(x_1-4)\\4(x_2-3)\end{bmatrix}$.

Now $g(X_1)=g(1,1)=(1-4)^2+2(1-3)^2-12=5>\epsilon=0.03$

Hence we linearize $g(X)$ about X_1 as follows to replace

$g(X)\le 0$ as $\quad g(X_1)+[\nabla g(X_1)]^T\begin{bmatrix}x_1-1\\x_2-1\end{bmatrix}\le 0$

or, $\quad 5+[-6,-8]\begin{bmatrix}x_1-1\\x_2-1\end{bmatrix}\le 0$

or, $\quad 5+(-6)(x_1-1)+(-8)(x_2-1)\le 0$

or, $\quad -6x_1-8x_2+19\le 0$

or, $\quad 6x_1+8x_2\ge 19$

We now consider the following *LPP* by adding the constraint $6x_1+8x_2\ge 19$ as

Maximize $\quad f=7-2x_1-4x_2$

subject to $\quad x_1+2x_2\le 6$

$6x_1+8x_2\ge 19$

$1\le x_1\le 6$

$1\le x_2\le 6$

The extreme point of the feasible region are

$A_1(1,13/8)$, $A_2(11/6,1)$, $B(4,1)$ and $C(1,5/2)$.

The values of the objective functions are

$f(1,13/8)=-3/2, f(11/6,1)=-2/3, f(4,1)=-5, f(1,5/2)=-5$

$\therefore$ The optimal solution of this *LPP* is $x_1=11/6$, $x_2=1$

$\therefore$ We take the next approximating point as $X_2=\begin{bmatrix}11/6\\1\end{bmatrix}$.

Now $g(X_2) = g(11/6,1) = \left(\frac{11}{6}-4\right)^2 + 2(1-3)^2 - 12 = \frac{25}{36} = 0.69 > \epsilon = 0.03$

We relinearize $g(X)$ about X_2 as follows and consider

$$g(X) \le 0 \text{ as } g(X_2) + [\nabla g(X_2)]^T \begin{bmatrix} x_1 - 11/6 \\ x_2 - 1 \end{bmatrix} \le 0$$

$$\text{or, } \frac{25}{36} + \left[-\frac{13}{3}, -8\right] \begin{bmatrix} x_1 - 11/6 \\ x_2 - 1 \end{bmatrix} \le 0$$

$$\text{or, } 156X_1 + 288X_2 \ge 599$$

We add this constraint to the previous *LPP* to get the following *LPP*

Maximize $f = 7 - 2x_1 - 4x_2$

subject to $x_1 + 2x_2 \le 6$

$6x_1 + 8x_2 \ge 19$

$156X_1 + 288X_2 \ge 599$

$1 \le x_1 \le 6$

$1 \le x_2 \le 6$

The extreme points of the feasible region are

$A_1(1,13/8)$, $B_1(17/12,21/16)$ and $C_1 = (311/156,1)$.

The values of the objective functions are

$$f(1,13/8) = -3/2, f(17/12,21/16) = -13/12, f(311/156,1) = -77/78$$

$\therefore$ The optimum solution is $(311/156,1)$.

We take $X_3 = \begin{bmatrix} 311/156 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.994 \\ 1 \end{bmatrix}$

Now $g(X_3) = g(1.994,1) = (1.994-4)^2 + 2(1-3)^2 - 12 = 0.026 < 0.03 = \epsilon$

Hence, the optimum solution is given by $X_1 = 1.994$, $X_2 = 1$.

106.20 SUMMARY

This module is devoted to some unconstrained and constrained numerical methods. Among unconstrained methods steepest descent method, Quadratically convergent method, Newton's method and Davidon-Fletcher-Powell method are discussed in detail. The constrained optimization methods that are considered here are Method of feasible direction, Zoutendijk's method, Rosen's gradient projection method and cutting plane method. All these methods are explained by examples.

106.21 SELF ASSESSMENT QUESTIONS

1. Using steepest descent method minimize the function
   $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 6x_1 - 4x_2 + 3x_3 + 9$ starting from the point (1, 2, -3).
2. Using steepest descent method minimize $f(x_1, x_2) = 2x_1 - x_2 + 8x_1^2 + 4x_1x_2 + x_2^2$ starting from the point (0,0).
3. Find the conjugate directions for the matrix $\begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}$.
4. Using Fletcher and Reeves method minimize $f(x_1, x_2) = x_1 - 2x_2 + 2x_1^2 + 4x_1x_2 + 4x_2^2$ starting from the point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
5. Using Davidon-Fletcher-Powell method
   minimize $f(x_1, x_2) = 8x_1^2 + 4x_2^2 - 24x_1 + 16x_2 + 35$ with $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ as the starting point.
6. Using Davidon-Fletcher-Powell method
   minimize $f(x_1, x_2) = 2x_1 - 3x_2 + 8x_1^2 + 12x_1x_2 + 9x_2^2$ with $\begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix}$ as the starting point.
7. Using Rosen's Gradient Projection method
   Minimize $f = 4x_1^2 + x_2^2 - 4x_1 - 4x_2$
   subject to $2x_1 + 4x_2 \le 5$
   $4x_1 + 3x_2 \le 6$

$x_1, x_2 \geq 0$

starting from the point $(1/2, 1)$.

8. Using cutting plane method

Maximize $f = 7 - 2x_1 - 4x_2$

subject to $(x_1 - 4)^2 + 2(x_2 - 3)^2 - 12 \leq 0$

$x_1 + 2x_2 - 6 \leq 0$

$1 \leq x_1, x_2 \leq 6$

with the tolerance as $\epsilon = 0.03$.

9. Using cutting plane method

Maximize $f = 1 - 4x_1 - 2x_2$

subject to $2(x_1 - 2)^2 + (x_2 - 3)^2 - 12 \leq 0$

$2x_1 + x_2 - 3 \leq 0$

$0 \leq x_1, x_2 \leq 5$

with $\epsilon = 0.2$.

106.22 SUGGESTED FURTHER READINGS

1. Optimization Theory and Applications : S.S. Rao, Wiley Eastern Limited.
2. Optimization methods in operations research and systems analysis: K.V. Mital & C. Mohan, New Age International Publishers.
3. Operations Research : Hamdy A. Taha, Prentice Hall of India Pvt. Ltd.
4. Non linear Programming Theory and Algorithms, M. Bazarra, M. Shrali & C. Shetty Wiley, New York.
5. Foundations of optimization : C. Beightler, D. Phillips & D. Wilde Prentice Hall.

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M.Sc. Course
in
Applied Mathematics with Oceanology
and
Computer Programming

PART-II

Paper-IX

Special Paper : OR

Module No. - 107

ADVANCED OPTIMIZATION AND OPERATIONS RESEARCH-I

(INTEGER PROGRAMMING)

Module Structure

107.1 Introduction

107.2 Objectives

107.3 Key Words

107.4 IPP at a Glance

107.4.1 Examples of IPP

107.5 Gomory Fractional (all Integers) Algorithm

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107.9 Self Assessment Questions

107.10 References

107.1 Introduction

We have dealt with problems wherein all basic variables (including slack or surplus variables) were permitted to take any non-negative real value. In many situations, fractional values are also there like 8.75 Kg of raw material, 3.52 man hours and 2.36 meter length of a sheet. However there are many problems, especially in business and industry, in which only integral values for the variables in optimal solutions make sense. For eample, it is not possible to run 8.6 buses on a route or to open 2.5 branhes of a bank or to employ 3.6 lathes in a workshop

A linear programming problem in which some or all of the variables must take non-negative integer values is ommonly referred to as integer linear programming problems.

When all the variables are integers, it is called an all integer programming problem.

In case only some of the variables are restricted to have integer values, the problem is said to be a mixed integer programming problem.

In some situations each variable can take on the values of either zero or one as in 'do' and 'not to do' type decisions; such problems are referred to as zero one programming problems.

107.2 Objectives

After going through this unit you will be able to learn about —

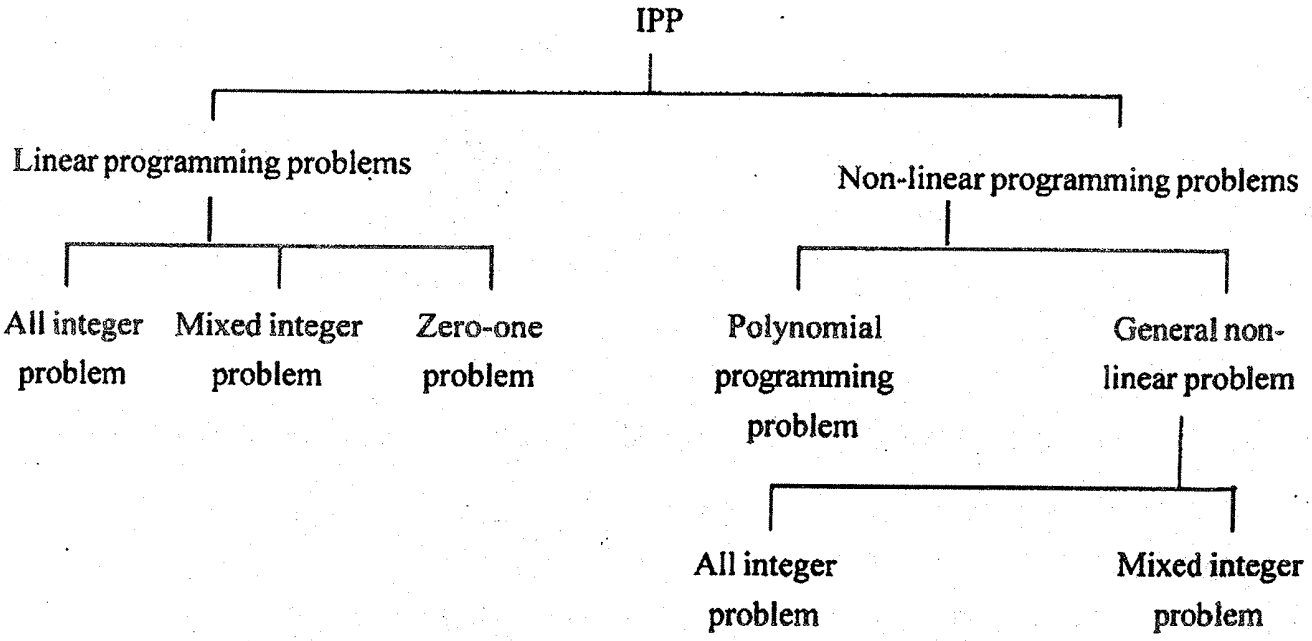
- What is integer programming problem?
- Examples of integer programming problem
- Gomory fractional algorithm
- Branch-and-Bound method

107.3 Key words

Integer programming problem (IPP), Gomory method, Branch-and-Bound method.

107.4 IPP at a Glance

Different types of IPP and their solutions methods are given below:



All integer problem and mixed integer problem are solved using

(i) Gomory cutting plane method.

and (ii) Branch-and-Bound method.

The zero-one problem and polynomial programming problem are solved by using

(i) Cutting plane method

(ii) Branch-and-Bound method

(iii) Balas method.

Non-linear all integer problem and mixed integer problem are solve by generalised penalty function method.

107.4.1. Examples of IPP

Example 4.1. (Assignment problem)

There are n jobs to be manufactured and n machines are available. Each job can be processed on each of the machines. The cost of processing is represented by the table below. If one job is to be processed on one machines only, the problem is to determine the optimal assignment policy that will turn out all jobs at minimum total cost. Formulate this problem as I.P. model.

Formulation as I.P. Problem

	Jobs			
Machines	C_{11}	C_{12}	...	C_{1n}
	C_{21}	C_{22}	...	C_{2n}
	⋮	⋮	...	⋮
	C_{n1}	C_{n2}	...	C_{nn}

Let $x_{ij} = \begin{cases} 1, \text{ if } i\text{th machine is assigned the } j\text{th job} \\ 0, \text{if } i\text{th machine is not assigned the } j\text{th job} \end{cases}$

Then the I.P. model is given by

$$\text{Minimize } z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\, x_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}\, x_{ij},$$

subject to constraints

$$\sum_{j=1}^{n} x_{ij} = 1, i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} x_{ij} = 1, j = 1, 2, \ldots, n.$$

Example 4.2. (Travelling Salesman Problem)

A book salesman who lives at city 1 must call once a month on four customers located in cities 2, 3, 4 and 5. The following table gives the distance in Kilometers among the different cities.

From city \ To city	1	2	3	4	5
1	–	210	150	250	110
2	210	–	100	80	130
3	150	100	–	60	105
4	250	80	60	–	90
5	110	130	150	90	–

The objective is to minimize the total distance travelled by the salesman. Formulate the problem as I.P. model.

Formulation as I.P. Problem:

Let $x_{ij} = \begin{cases} 1, \text{ if the salesman travels from city } i \text{ to city } j; \\ 0, \text{ otherwise.} \end{cases}$

The necessary condition for a complete route is that city i connects to one city only and that city j is reached from exactly one city. The I.P. model for the problem is

$$\text{Minimize } z = [210x_{12} + 150x_{13} + 250x_{14} + 110x_{15} + 210x_{21}$$

$$+ \ldots + 90x_{54} + M(x_{11} + x_{22} + x_{33} + x_{44} + x_{55})]\,Km$$

where M is a prohibitively high cost.

Subject to,

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = 1$$

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 1$$

$$\vdots \quad \vdots \quad \vdots$$

$$x_{51} + x_{52} + x_{53} + x_{54} + x_{55} = 1$$

$$x_{11} + x_{21} + x_{31} + x_{41} + x_{51} = 1$$

$$\vdots \quad \vdots \quad \vdots$$

and $x_{15} + x_{25} + x_{35} + x_{45} + x_{55} = 1$.

Example 4.3 (Capital Budgeting Problem)

A Company manufacturing chemicals has 4 independent investment projects and must allocate a fixed capital budget to one or more of them so that the Comp 's total assets are maximized. The estimated investments and the anticipated cash outflows associated with these projects are given in the table below:

Project	Investment (Rs. lakhs)		Cash in flows (Rs. lakhs)
	1st year	2nd year	
A	60	160	105
B	108	140	140
C	200	150	80
D	90	70	100

The Company has earmarked Rs. 600 lakhs for investment in the first year and Rs. 700 lakhs in the second year. If project A and C are mutually exclusive, how should the investment be made so that the Company's total assets are maximized?

Formulation as I.P. Problem

$$x_j = \begin{cases} 1, \text{ if there is investment on the } j\text{th project} \\ 0, \text{ otherwise (j=1,2,3,4)} . \end{cases}$$ for projects A, B, C, D.

Then the objective is

Maximize $z = 105x_1 + 140x_2 + 80x_3 + 100x_4$

subject to $60x_1 + 108x_2 + 200x_3 + 90x_4 \leq 600$

$160x_1 + 140x_2 + 150x_3 + 70x_4 \leq 700$

Since projects A and C are mutually exclusive $x_1 + x_3 = 1$.

107.5. Gomory Fractional (all Integers) Algorithm

Gomory cutting plane algorithm starts solving the continuous L.P. problem.

We consider a table which contains (m+n) variables of which $x_i (i = 1,2,...,m)$ are basic and $y_j (j = 1,2,...,n)$ are non-basic. Some or all of these basic variables have non-integer values.

Basic	x_1	x_2	...	x_i	...	x_m	y_1	y_2	...	y_j	...	y_n	b
x_1	1	0	...	0	...	0	a_{11}	a_{12}	...	a_{1j}	...	a_{1n}	b_1
x_2	0	1	...	0	...	0	a_{21}	a_{22}	...	a_{2j}	...	a_{2n}	b_2
⋮								⋮					⋮
x_i	0	0	...	1	...	0	a_{i1}	a_{i2}	...	a_{ij}	...	a_{in}	b_i
⋮								⋮					⋮
x_m	0	0	...	0	...	1	a_{m1}	a_{m2}	...	a_{mj}	...	a_{mn}	b_m
$\bar{c}_j$	0	0	...	0	...	0	$\bar{c}_1$	$\bar{c}_2$	...	$\bar{c}_j$	...	$\bar{c}_n$	

The basic variable with the largest fractional value is selected.

Let this basic variable be x_i. In case of tie, choose the variable with lowest contribution for a maximization problem or the highest cast for a minimization problem. The ith constraint equation can be expressed as

$$x_i + a_{i1}y_1 + a_{i2}y_2 + \ldots + a_{ij}y_j + \ldots + a_{in}y_n = b_i$$

or, $$x_i + \sum_{j=1}^{n} a_{ij}y_j = b_i$$

or, $$x_i = b_i - \sum_{j=1}^{n} a_{ij}y_j, b_i \text{ non-integer.}$$

Let $b_i = (b_i)_i + f_i$ and $a_{ij} = (a_{ij})_i + f_{ij}$

where $(b_i)_i$ and $(b_{ij})_i$ are integer part of b_i and a_{ij} and f_i and and f_{ij} are fraction part of b_i and a_{ij} respectively.

Here $0 < f_i, f_{ij} < 1$.

Thus $$x_i = (b_i)_i + f_i - \sum_{j=1}^{n} \{(a_{ij})_i + f_{ij}\} y_j$$

$$= (b_i)_i + f_i - \sum_{j=1}^{n} (a_{ij})_i y_j - \sum_{j=1}^{n} f_{ij} y_j$$

$$\therefore f_i - \sum_{j=1}^{n} f_{ij} y_j = x_i - (b_i)_i + \sum_{j=1}^{n} (a_{ij})_i y_j.$$

Since R.H.S. is an integer (+ve, –ve, 0), therefore L.H.S. is also an integer.

Since $0 < f_i < 1$ and also $0 \le f_i \le 1$ we have

$$f_i - \sum_{j=1}^{n} f_{ij} y_j \le 0$$

or, $f_i - \sum_{j=1}^{n} f_{ij} y_j \le 0$ is called Gomories constraint.

or, $\sum_{j=1}^{n} f_{ij} y_j \ge f_i$

or, $\sum_{j=1}^{n} f_{ij} y_j - G = f_i$ where G is surplus variable.

or, $-\sum_{j=1}^{n} f_{ij} y_j + G = -f_i$.

This is the Gomories constraint equation.

Example 5.1. Maximize $z = 7x_1 + 9x_2$

Subject to $-x_1 + 3x_2 \le 6$

$7x_1 + x_2 \le 35$

$x_1, x_2 \ge 0$ and integers.

Solution. Ignoring integer restriction, the problem can be expressed in standard LPP as

Maximize $z = 7x_1 + 9x_2 + 0.x_3 + 0.x_4$

subject to $-x_1 + 3x_2 + x_3 = 6$

$7x_1 + x_2 + x_4 = 35$

$x_1, x_2, x_3, x_4 \ge 0,$

where x_3 and x_4 are slack variables.

As $z_j - c_j \ge 0$ for all j, optimality condition has been satisfied.

Using simplex method

		c_j	7	9	0	0	
V_B	C_B	b	Y_1	Y_2	Y_3	Y_4	Min ratio
Y_3	0	6	−1	[3]	1	0	6/3 = 2→
Y_4	0	35	7	1	0	0	35/1 = 35
$z_j - c_j$			−7	−9↑	0	0	
Y_2	9	2	$-1/3$	1	$1/3$	0	
Y_4	0	33	[$22/3$]	0	$-1/3$	1	→
$z_j - c_j$			−10↑	0	3	0	
Y_2	9	$7/2$	0	1	$7/22$	$1/22$	
Y_1	7	$9/2$	1	0	$-1/22$	$3/22$	
$z_j - c_j$			0	0	$28/11$	$15/11$	

As $z_j - c_j \geq 0$ for all j, optimality condition ahs been satisfied.

The solution is $x_1 = 9/2, x_2 = 7/2$ and $Z_{max} = 63$.

The solution obtained is optimal feasible non-integer solution.

To construct Gomory's constraint, we can select any row since the fractional part is equal $\left(\frac{1}{2}\right)$ is both the rows (select the larger fraction part of b).

We consider first row,

$$b_1 = \frac{7}{2} = 3\frac{1}{2} = 3 + \frac{1}{2}$$

$$\frac{7}{22} = 0 + \frac{7}{22}, \frac{1}{22} = 0 + \frac{1}{22}.$$

Gomory's constraint will be

$$(0+0)x_1+(1+0)x_2+\left(0+\frac{7}{22}\right)x_3+\left(0+\frac{1}{22}\right)x_4=3+\frac{1}{2}$$

or, $$-\frac{7}{22}x_3-\frac{1}{22}x_4\le-\frac{1}{2}$$

or, $$-\frac{7}{22}x_3-\frac{1}{22}x_4+G_1=-\frac{1}{2}$$

Introducing Gomory's constraint in the optimal table.

		c_j	7	9	0	0	0
V_B	C_B	b	Y_1	Y_2	Y_3	Y_4	G_1
Y_2	9	$7/2$	0	1	$7/22$	$1/22$	0
Y_1	7	$9/2$	1	0	$-1/22$	$3/22$	0
G_1	0	$-1/2$	0	0	$-7/22$	$-1/22$	1
z_j-c_j			0	0	$56/22$	$15/11$	0

Since basic variable G_1 is negative and z_j-c_j is either zero or positive, dual simplex method can be applied to find the new optimal solution.

		c_j	7	9	0	0	0
V_B	C_B	b	Y_1	Y_2	Y_3	Y_4	G_1
Y_2	9	$7/2$	0	1	$7/22$	$1/22$	0
Y_1	7	$9/2$	1	0	$-1/22$	$3/22$	0
G_1	0	$-1/2$	0	0	$-7/22$	$-1/22$	1
z_j-c_j			0	0	$28/11$	$15/11$	0
Replacement ratio			–	–	$\frac{28/11}{-7/22}=-8\uparrow$	$\frac{15/11}{-1/22}=-30$	–

$\frac{z_j - c_j}{y_{rj}}, y_{rj} < 0$							
Y_2	9	3	0	1	0	0	1
Y_1	7	$4\frac{4}{7}$	1	0	0	$\frac{1}{7}$	$-\frac{1}{7}$
Y_3	0	$1\frac{4}{7}$	0	0	1	$\frac{1}{7}$	$-\frac{22}{7}$
$z_j - c_j$			0	0	0	1	8

As $z_j - c_j \geq 0$, optimility condition has been satisfied.

$x_1 = 4\frac{4}{7}$, $x_2 = 3$ and $z_{max} = 59$, but, x_1 is not an integer.

We consider the 2nd row and construct the Gomory's constraint.

$$-\frac{1}{7}x_4 - \frac{6}{7}G_1 \leq -\frac{4}{7} \quad \left[(0+0)x_1 + (0+0)x_2 + (0+0)x_3 + \left(1+\frac{1}{7}\right)x_4 + \left(-1+\frac{6}{7}\right)G_1 \leq 4 + \frac{4}{7}\right]$$

or, $-\frac{1}{7}x_4 - \frac{6}{7}G_1 + G_2 = -\frac{4}{7}$.

The modified table is

		c_j	7	9	0	0	0	0	
V_B	C_B	X_B	Y_1	Y_2	Y_3	Y_4	G_1	G_2	
Y_2	9	3	0	1	0	0	1	0	
Y_1	7	$4\frac{4}{7}$	1	0	0	$\frac{1}{7}$	$-\frac{1}{7}$	0	
Y_3	0	$1\frac{4}{7}$	0	0	1	$\frac{1}{7}$	$\frac{1}{7}$	0	
G_2	0	$-\frac{4}{7}$	0	0	0	$\boxed{-\frac{1}{7}}$	$-\frac{6}{7}$	1	→
$z_j - c_j$			0	0	0	1↑	8	0	
Replacement ratio			–	–	–		$\frac{1}{-1/7} = -7$	$\frac{8}{-6/7} = \frac{28}{3}$	–
$\frac{z_j - c_j}{y_{4j}}, y_{4j} < 0$									

Y_2	9	3	0	1	0	0	1	0
Y_1	7	4	1	0	0	0	−1	1
Y_3	0	1	0	0	1	0	−4	1
Y_4	0	4	0	0	0	1	6	−7
$z_j - c_j$			0	0	0	0	2	7

All x_i's are non-negative integer, hence optimal solution is $x_1 = 4, x_2 = 3$ and $z_{max} = 55$.

Examples 5.2 Solve the following IPP using Gomory's cutting plane algorithm

Maximize $z = x_1 + x_2$

subject to $3x_1 + 2x_2 \le 5$

$x_2 \le 2$

$x_1, x_2 \ge 0$ and are integer

Sol. max $z = x_1 + x_2 + 0.x_3 - 0.x_4$

subject to $3x_1 + 2x_2 + x_3 = 5$

$x_1 + x_4 = 2$

$x_1, x_2, x_3, x_4 \ge 0.$

		c_j	1	1	0	0	
V_B	C_B	X_B	Y_1	Y_2	Y_3	Y_4	
Y_3	0	5	[3]	2	1	0	$5/3$
Y_4	0	2	0	1	0	1	–
$z_j - c_j$			−1↑	−1	0	0	
Y_1	1	$5/3$	1	$2/3$	$1/3$	0	$5/2$
Y_4	0	2	0	[1]	0	1	$2/1 \rightarrow$
$z_j - c_j$			0	$-1/3$ ↑	$1/3$	0	
Y_1	1	$1/3$	1	0	$1/3$	$-2/3$	
Y_2	1	2	0	1	0	1	
$z_j - c_j$			0	0	$1/3$	$1/3$	

Optimal solution is $x_1 = 1/3, x_2 = 2, z_{max} = 7/2$.

Here

$$(1+0)x_1 + (0+0)x_2 + \left(0+\frac{1}{3}\right)x_3 + \left(-1+\frac{1}{3}\right)x_4 = (0+1/3)$$

or,

$$-\frac{1}{3}x_3 - \frac{1}{3}x_4 \le -\frac{1}{3} \quad \text{or,} \quad -\frac{1}{3}x_3 - \frac{1}{3}x_4 + G = -1/3$$

Introducing the Gomories constraint in the optimal table.

	c_j	1	1	0	0	0	
V_B	C_B	X_B	Y_1	Y_2	Y_3	Y_4	G_1	
Y_1	1	$1/3$	1	0	$1/3$	$-2/3$	0	
Y_2	1	2	0	1	0	1	0	
G_1	0	$-1/3$	0	0	$\boxed{-1/3}$	$-1/3$	1	$\rightarrow$
$z_j - c_j$			0	0	$1/3 \uparrow$	$1/3$	0	
Replacement ratio			–	–	$\frac{1/3}{-1/3} = -1$	$\frac{1/3}{-1/3} = -1$		
Y_1	1	0	1	0	0	−1	1	
Y_2	1	2	0	1	0	1	0	
Y_3	0	1	0	0	1	1	−3	
$z_j - c_j$			0	0	0	0	1	

Solution is $x_1 = 0, x_2 = 2, z_{max} = 2$. This also satisfies the integer requirement. Hence this is the optimal integer solution.

107.6 Branch-and-Bound Method

Branch-and Bound method is the most widely used method for solving pure as well as mixed integer problems. Most commercial computer codes use this method for solving I.P.P.

In this method, the problem is first solved as a continuous L.P.P. ignoring the integrality condition. If in the optimal solution some variable, say, x_j is not an integer, then

$$x_j^* < x_j < x_{j+1}^*$$

where x_j^* and x_{j+1}^* are consecutive non-negative integers.

It follows that any feasible integer value of x_j must satisfy one of the two conditions

$$x_j \le x_j^* \quad \text{or} \quad x_j \ge x_{j+1}^* .$$

These two conditions are mutually exclusive and when applied separately to the L.P.P., form two different subproblems. Thus the original problem is 'branched' into two subproblems.

Geometrically, it means that the branching process eliminates that portion of the feasible region that contains no feasible integer solution.

This branching process yields two subproblems, one by adding the constraint $x_j \le x_j^*$ and the other by adding $x_j \ge x_{j+1}^*$ to the original set of constraints. Each of these sub problems is then solved separately as a linear program, using the same objective function of the original problem. If any sub problem yields an optimal integer solution, it is not further branched. However, if it yields a non-integer solution it is further branched into two sub problems.

This process is continued, until each problem terminates with either integer valued optimal solution or there is evidence that it cannot yield a better one.

Example 6.1. Solve the problem using branch-and-bound method

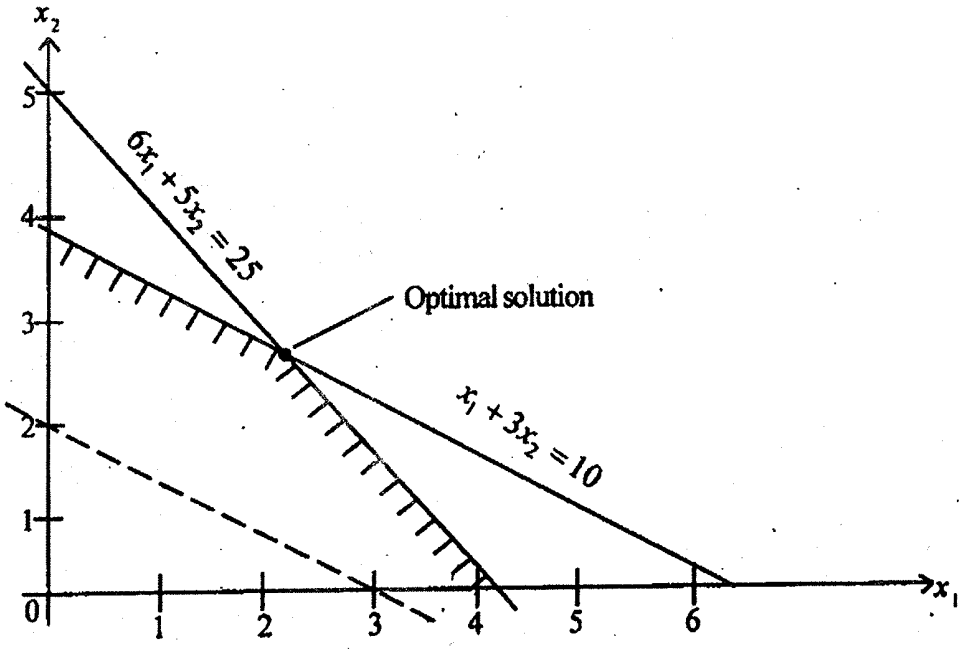
Maximize $z = 2x_1 + 3x_2$

subject to $6x_1 + 5x_2 \le 25$

$x_1 + 3x_2 \le 10$

x_1, x_2 non-negative integers.

Solution. The given problem is solved as a continuous linear programming problem ignoring the integrality condition.



The corner points of the feasible region are (0,0), (0,10/3), (25/6, 0), (25/13, 35/13).

The optimal solution is $x_1 = \frac{25}{13}, x_2 = \frac{35}{13}$,

$z_{max} = \frac{155}{13} \simeq 12.$

Since the solution is non-integer with both x_1 and x_2 having fractional values, and variable may be arbitrarily selected for branching.

Now, $x_2 = \frac{35}{13}$ gives $2 < x_2 < 3$.

Thus adding a new constraint $x_2 \leq 2$ or $x_2 \geq 3$ two sub problems are

Sub problem 1

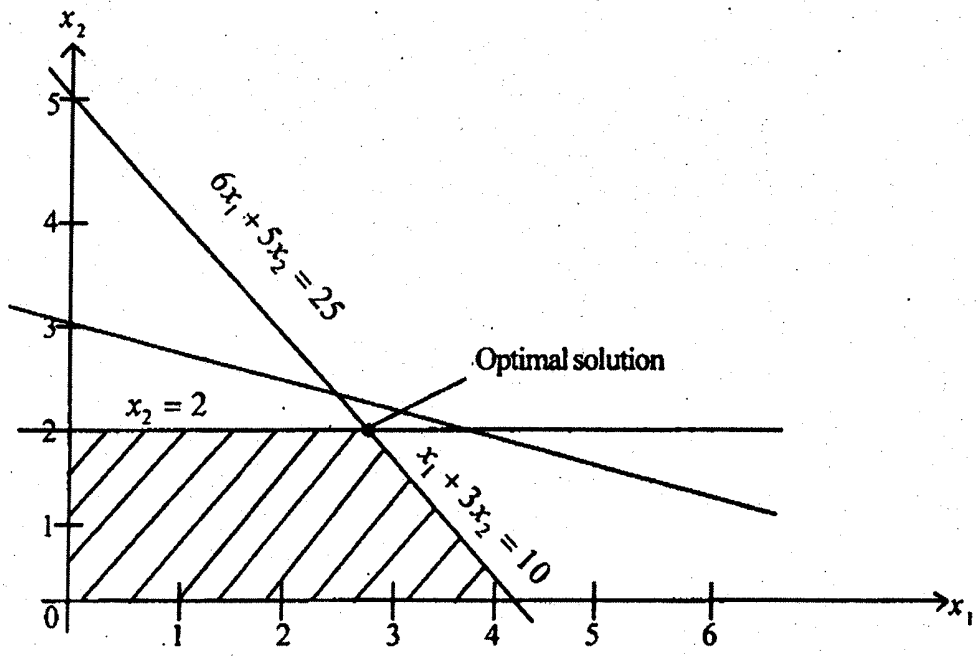
$$\max z = 2x_1 + 3x_2$$

subject to $6x_1 + 5x_2 \le 25$

$x_1 + 3x_2 \le 10$

$x_2 \le 2$

x_1, x_2 non-negative integers



The optimal solution is

$$x_1 = 2.5, x_2 = 2, z_{max} = 11$$

Sub problem 2.

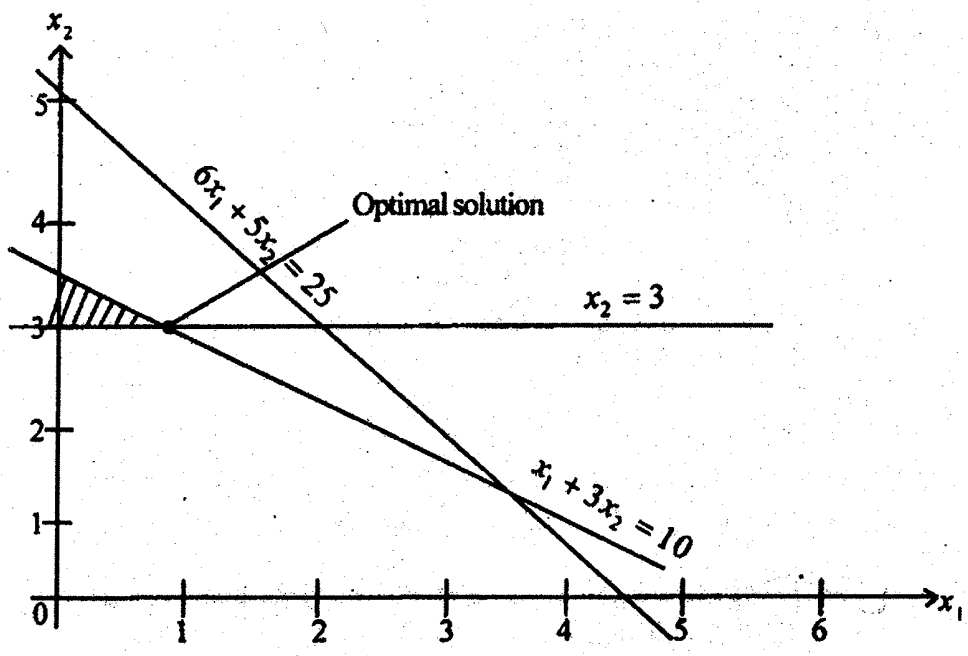
$$\max z = 2x_1 + 3x_2$$

subject to $6x_1 + 5x_2 \le 25$

$x_1 + 3x_2 \leq 10$

$x_2 \leq 3$

x_1, x_2 non-negative integers



The optimal solution is $x_1 = 1, x_2 = 3, z_{max} = 11$.

Solution of sub problem z is a feasible integer solution to the given problem. Since x_1, x_2 are integers, there is no need to branch sub problem 2 further. Sub problem 1 can be branch to look for alternative integer solution.

$x_1 = 2.5$ gives $2 < x_1 < 3$.

Adding the constraints $x_1 \leq 2$ or $x_1 \geq 3$ to the sub problem 1, the sub problem 1 can be further divided.

Sub problem 3

$$\max z = 2x_1 + 3x_2$$

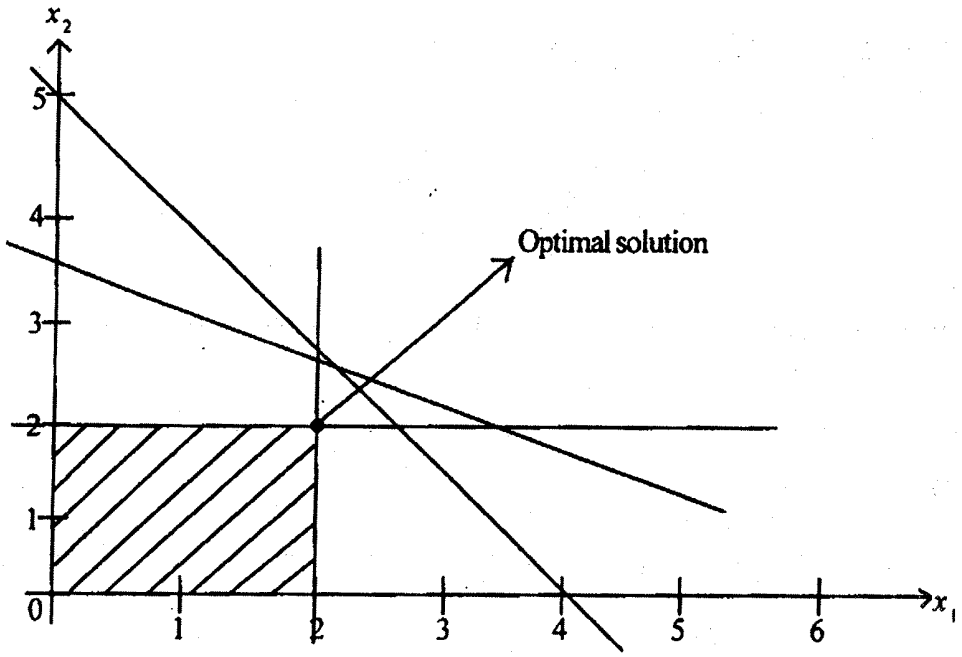
subject to $6x_1 + 5x_2 \le 25$

$x_1 + 3x_2 \le 10$

$x_2 \le 2$

$x_1 \le 2$

x_1, x_2 non-negative integers



$$x_1 = 2, x_2 = 2, z_{max} = 10$$

Sub problem 4

$$\max z = 2x_1 + 3x_2$$

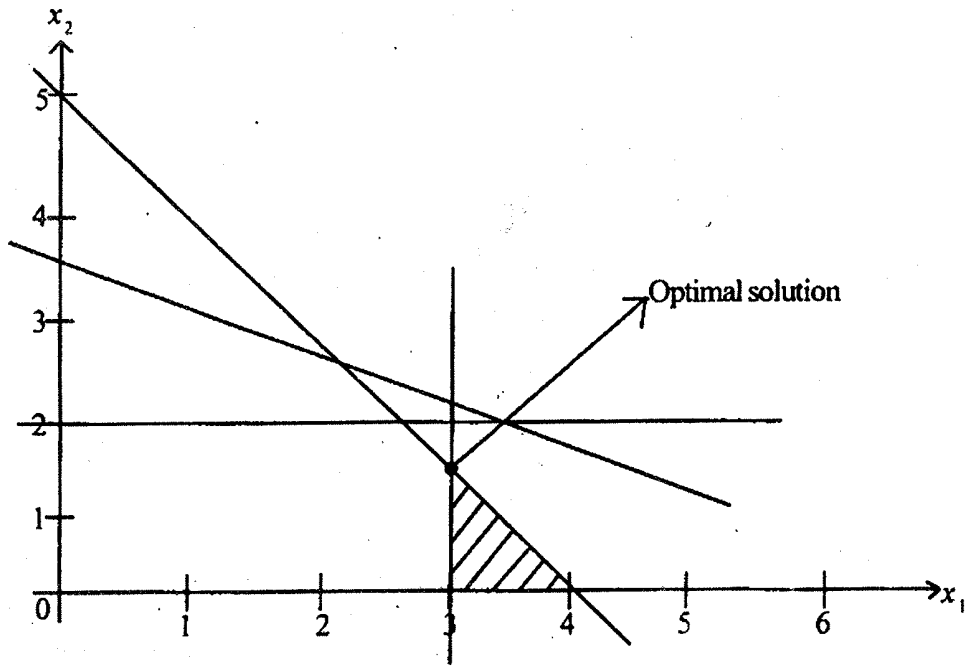
subject to $6x_1 + 5x_2 \le 25$

$x_1 + 3x_2 \le 10$

$x_1 \leq 2$

$x_1 \geq 3$

x_1, x_2 non-negative integers



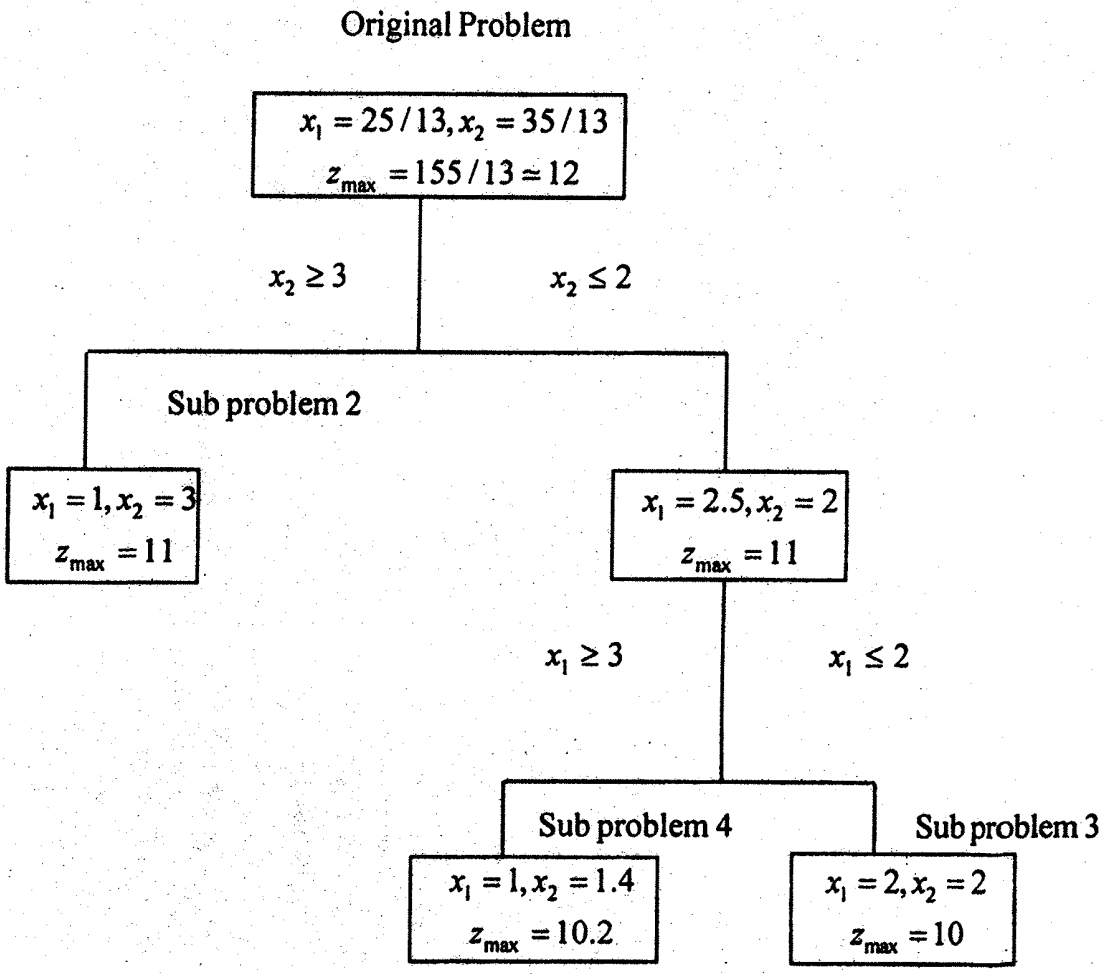
$x_1 = 3, x_2 = 1.4, z_{max} = 10.2$

Since the solution is non-integer, sub problem 4 can be further branched with x_2 as the branching variable. But, value of objective function ($z_{max} = 10.2$) is inferior to the lower bound ($z_{max} = 11$) and hence this does not promise a solution better than the one already obtained.

Thus solution of sub problem 2 is the best available integer optimal solution.

$x_1 = 1, x_2 = 3, z_{max} = 11$

Diagramatic representation



Example 6.2. Solve the following all-integer programming problem using branch-and-bound method.

$$\max z = 3x_1 + 2.5x_2$$

subject to $x_1 + 2x_2 \geq 20$

$3x_1 + 2x_2 \geq 50$

$x_1, x_2 \geq 0$ and integers.

Solution. Solution obtained by graphical method is

$x_1 = 15, x_2 = 2.5$ and $z_{min} = 51.25$

Here $x_2 = 2.5$, $2<2.5<3$.

New constraints are $x_2 \le 2$, $x_2 \ge 3$.

Sub problem 1

$$\max z = 3x_1 + 2.5x_2$$

subject to $x_1 + 2x_2 \ge 20$

$3x_1 + 2x_2 \ge 50$

$x_2 \le 2$

$x_1, x_2 \ge 0$ and integers.

Solution: $x_1 = 16, x_2 = 2, z_{min} = 53$

Sub problem 2

$$\max z = 3x_1 + 2.5x_2$$

subject to $x_1 + 2x_2 \ge 20$

$3x_1 + 2x_2 \ge 50$

$x_2 \ge 3$

$x_1, x_2 \ge 0$ and integers.

Solution: $x_1 = 14.66, x_2 = 3, z_{min} = 51.5$

Sub problem 2.1

$$\max z = 3x_1 + 2.5x_2$$

subject to $x_1 + 2x_2 \ge 20$

$3x_1 + 2x_2 \ge 50$

$x_2 \ge 3$

$x_1 \le 14$

$x_1, x_2 \ge 0$ and integers.

Solution: $x_1 = 14, x_2 = 4, z_{min} = 52$

Sub problem 2.2

$$\max z = 3x_1 + 2.5x_2$$

subject to $x_1 + 2x_2 \ge 20$

$3x_1 + 2x_2 \ge 50$

$x_2 \ge 3$

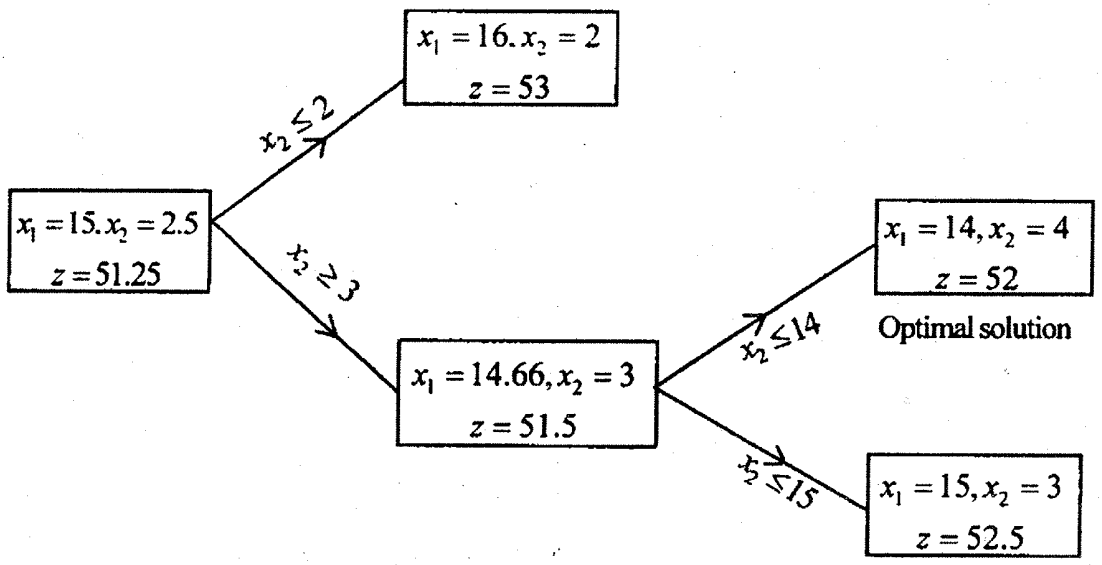
$x_1 \ge 15$

$x_1, x_2 \ge 0$ and integers.

Solution: $x_1 = 15, x_2 = 3, z_{min} = 52.5$

Minimum value occur at subproblem 2.1.

Therefore the optimal solution is $x_1 = 14, x_2 = 4$ and $z_{min} = 52$.



107.7. Zero-One Programming

The linear programming problem in which all the variables are constrained to take values of 0 or 1 only, is called a zero-one (or binary) L.P. problem. The assignment problem, problems of capital budgeting, project selection, facility location, machine/job scheduling etc. involve zero-one integer solutions.

The zero-one integer programming problem can be stated as :

Minimize $Z = \sum_{j=1}^{n} c_j x_j$

subject to the constraints

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i, i = 1, 2, ..., n$$

and $x_j = 0$ or 1.

Example 7.1. (Capital budgeting problem).

A corporation is considering four possible investment opportunities.

Project	Present value of expected return (Rs. in thousand)	Capital required year wise project		
		Year 1	Year 2	Year 3
1	650	700	550	400
2	700	850	550	350
3	225	300	150	100
4	250	350	200	-
Capital available for investment		1200	700	400

Projects 1 and 2 are mutually exclusive and project 4 is contingent on the prior acceptence of project 3. Formulate an integer programming model to determine which projects should be accepted and which should be rejected to maximize the present value from accepted projects.

Model formulation

$$\text{Let } x_j = \begin{cases} 1, \text{ if project } j \text{ is accepted} \\ 0, \text{if project } j \text{ is rejected} \end{cases}$$

Integer LP model is

Maximize (Total present value) $Z = 650x_1 + 700x_2 + 225x_3 + 250x_4$

Subject to

$$700x_1 + 850x_2 + 300x_3 + 350x_4 \leq 1200$$

$$550x_1 + 550x_2 + 150x_3 + 200x_4 \leq 700$$

$$400x_1 + 350x_2 + 100x_3 \leq 400$$

$$x_1 + x_2 \geq 1$$

$$x_4 - x_3 \leq 1$$

and $x_j = 0$ or 1.

107.8 Summary

Two important methods, viz. Gomory cutting plane and branch-and-bound are described here to solved an integer programming problem. Both the methods are illustrated by several examples. An exercise is also provided at the end of this module for self study.

107.9. Self assessment questions

1. What do you mean by integer programming problem? Give some examples of it.
2. Explain Gomory cutting plane method to solve an IPP.
3. Explain branch-and-bound method to solve an IPP.
4. Solve the following IPP using branch-and-bound method.

   Maximize $z = x_1 + x_2$

   subject to $3x_1 + 2x_2 \leq 12$

   $x_2 \leq 2$

   $x_1, x_2 \geq 0$ and integers.

5. Use branch-and-bound method to solve the following IPP.

   Maximize $z = 7x_1 + 9x_2$

   subject to $-x_1 + 3x_2 \leq 6$

   $7x_1 + x_2 \leq 35$

   $x_1, x_2 \geq 0$ and integers.

6. Solve the following LPP graphically and discuss the effect of rounding off the variables to the nearest integers

Maximize $z = 3x_1 + 10x_2$

subject to $x_1 + 5x_2 \le 12$

$x_1 \le 3$

$x_1, x_2 \ge 0$. [Ans. $x_1 = 2, x_2 = 2, Z_{max} = 26$]

7. Solve the following integer linear programming problem by using Gomory fractional cut :

Maximize $z = -4x_1 + 5x_2$

subject to $-3x_1 + x_2 \le 6$

$2x_1 + 4x_2 \le 12$

x_1, x_2 are non-negative integers. [Ans. $x_1 = 0, x_2 = 2, Z_{max} = 10$]

8. The non-integer optimal solution of an LPP is given below. Find the all integer optimal solution:

Basis	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	b_i
Y_1	1	0	0	$3/10$	$1/5$	0	$5/2$
Y_2	0	1	0	$1/20$	$1/5$	0	$5/4$
Y_3	0	0	1	$1/4$	0	1	$25/4$

where Y_4, Y_5, Y_6 are either slack or surplus variables.

[$x_1 = 2, x_2 = 1, x_3 = 6, Z_{max} = 26$]

9. Solve the following problem by branch-and-bound technique:

Maximize $z = x_1 + x_2$

subject to $3x_1 + 2x_2 \le 12$

$x_2 \le 2$

$x_1, x_2 \ge 0$ and are integers. [Ans. $x_1 = 2, x_2 = 2, Z_{max} = 4$]

10. Maximize by the branch and bound technique

$z = 7x_1 + 9x_2$

subject to $-x_1 + 7x_2 \le 6$

$7x_1 + x_2 \le 35$

$0 \le x_1, x_2 \le 7$

x_1, x_2 are integers. [Ans. $x_1 = 4, x_2 = 3, Z_{max} = 55$]

11. Solve the following problem

Maximize $z = 3x_1 + 3x_2 + 13x_3$

subject to $-3x_1 + 6x_2 + 7x_3 \le 8$

$5x_1 - 3x_2 + 7x_3 \le 8$

$0 \le x_j \le 5$

all x_j are integers for $j = 1, 2, 3$.

107.10. References :

1. P.K. Gupta and D.S. Hira, Problems in Operations Research, S. Chand & Company Ltd.
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3. H.A. Taha, Operations Research : An Introduction, Prentice Hall Inc., New Jersey.
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M.Sc. Course

in

Applied Mathematics with Occanology

and

Computer Programming

PART - II

PAPER - IX (OR) **Module No - 108** GROUP -

Content :

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108.1 Introduction

Linear programming problem has only one objective viz maximization of profit or minimization of cost. But in most of todays problems profit maximization is not the only object of management. In fact, now-a-day firms needs to obey many social responsibilities like public relations, labour relations, protection of environments etc. These situations generate problems having a set of goals with some given priorities. The solution of such problems does not satisfy all the goals fully. The

most important goal is tried to be achieved first then it is tried to achieve the second priority goal and continues until the achievement of all less important goals. Here an important point is to be noted that in the answer of such problem fulfillment of all goals may not be possible as the goals are in general of conflicting nature. Thus compromise solution is to be obtained for such problems and those problems are known as goal programming problems.

108.2 Objective

The linear programming problems deals with only one objective function and is over simplification of reality. In fact, most of the real life problems have multiobjectives of conflicting nature in general. These problems are called goal programming problems. In this module concept of goal programming is introduced first. Then graphical solution is found for different types of goal programming problems Finally, self assessment exercise is given.

108.3 Concept of Goal Programming Problem

Most real life situations involve problems with multiple goals and subgoals. Various such goals are expressed in different units of measurements such as rupees, hours, kilograms etc. Often multiple goals of management are in conflict or are achievable only at the expense of other goals. Further these goals are found to be incommensurable. So the solution process of such problems requires a technique where lower order goals are considered only after the higher-order goals are satisfied.

In goal programming we do not maximize or minimize the objective function directly as in linear programming problem. Here deviations between the goals and what can be achieved within the given set of constraints are minimized. So in goal programming problem the objective functions contains primarily the deviational variables. The important characteristic of goal programming problem is that it allows ordinal solution. The manager is usually called upon to specify the priority of the desired attainment of each goal or subgoal and rank them in ordinal sequence for decision analysis. Let there be n decision variables, m goals and N priorities.

A general goal programming problem contains two types of variables

i) decision variables $x_1, x_2, \ldots\ldots, x_n$.

ii) deviational variables $d_1^+, d_2^+, \ldots\ldots, d_m^+$ representing the amount of over-achievement of the 1st, 2nd, ..., mth goal respectively.

iii) deviational variables d_1^-, d_2^-,....., d_m^- representing the amount of under-achievement of the 1st, 2nd, ..., mth goal respectively.

A goal programming problem contains two types of constraints

i) structural constraints (not directly related to goals) and

ii) goal constraints (directly related to goals)

The general goal programming model is given by

$$\text{Minimize } z = \sum_{i=1}^{m}\sum_{k=1}^{N} P_k\left(w_{ik}^+ d_i^+ + w_{ik}^- d_i^-\right)$$

$$\text{subject to } \sum_{j=1}^{n} a_{ij}x_j + d_i^- - d_i^+ = b_i, \quad i=1, 2, ..., m$$

$$\sum_{j=1}^{n} a_{ij}x_j \leq = \geq b_i, \quad i=m+1, m+2, ..., m+p$$

$$x_j,\ d_i^+,\ d_i^- \geq 0,\ j=1, 2,...., n,\ i=1, 2,...., m$$

where P_k = the priority coefficient for the kth priority

w_{ik}^+ = the relative weight of the d_i^+ variable in the kth priority level

w_{ik}^- = the relative weight of the d_i^- variable in the kth priority level.

Here we note that there are m goals and the goals are ranked from the most important goal (Goal 1) to the least important goal (Goal m). The objective function coefficient for the deviational variables associated with the kth priority is P_k. It is assumed that $P_1 >> P_2 >> P_3 >>.....>> P_N$ where the symbol "$>>$" means "more important than". In fact P_1, P_2, ..., P_N are not assigned any actual values. Rather they are a convenient way of indicating that one goal is more important than the other. These coefficients indicate that weight of goal 1 is much larger than weight 2, which in turn is much larger than Goal 3 and so on. A lower priority goal can never be achieved at the expense of higher priority goal.

At the same priority level, the deviational variables may be given different weights w_{ik}^+, w_{ik}^- in the objective function so that deviational variables within the same priority have different weights.

Since both under and over-achievement of a goal cannot be achieved simultaneously, either one or both of them will be equal to zero i.e. $d_i^- \times d_i^+ = 0$ In other words, if either assumes a positive value, the other must be zero, and vice versa.

The decision maker must analyse each one of the m goals in terms of whether under or over achievement of the goal is satisfactory. If over-achievement is a acceptable, d_i^+ (similar to the surplus variable in LPP) can be removed from the objective function. On the other hand, if under-achievement is acceptable, d_i^- (similar to the slack variable in LPP) can be removed from the objective function. If exact achievement of the goal is derived, both d_i^- and d_i^+ must be included in the objective function, and ranked according to their permeative priority factors from the most important to the least important.

A lower priority goal is never sought to be achieved at the expense of higher priority goal. Thus the highest priority goal is considered first and an attempt is made to achieve it as closely as possible (often it is not achieved fully). The next priority goal is considered only when an optimum solution with respect to the first priority goal is achieved. Similarly, the next remaining goals are considered only when the optimality with respect to the higher priority goals is obtained. We continue in this manner until all the goals and priorities are considered.

108.4 Relationship between GPP and LPP

We know that a LPP has only one objective. So to get the relationship between LPP and GPP we have to consider a problem with only one goal. The following example serves this purpose.

108.4.1 Example

Let a company has to produce two products (e.g. copper rod & iron rod) A and B each product requires time in two departments 1 and 2. Product A requires 30 hrs in department 1 and 20 hrs. in department 2. Product B requires 15 hrs. in department 1 and 50 hrs. in department 2. Production time is limited in department 1 to 60 hrs. and is department 2 to 100 hrs. Contributions to profits for the two products are Rs 55 and Rs 45 respectively.

Managements goal is to minimize the profit.

Solution

We first express the problem as a LPP. Let x_1 and x_2 be the number of units of product A and B respectively. The LP formulation is

Maximize $z = 55x_1 + 45x_2$ .....(4.1)

subject to constraints $30x_1 + 15x_2 \leq 60$ .....(4.2)

$$20x_1 + 50x_2 \leq 100 \quad \text{.....(4.2)}$$

$$x_1, x_2 \geq 0 \quad \text{.....(4.3)}$$

Using simplex method we get the solution of this LPP as $x_1 = \frac{5}{4}$, $x_2 = \frac{3}{2}$ and $z_1 = 136\frac{1}{4}$.

To get the goal programming formulation we note that the problem is a maximization problem. We are to maximize the profit. So we set an arbitrary high goal, say Rs 1500. We need two non-negative deriational variables d^+ and d^- for the goal. The deviational variable d^+ represents the amount by which we over-achieve the profit goal Rs 1500 whereas the deviational variable d^- represents the amount by which we under-achieve the profit goal Rs 1500. As the goal Rs 1500 of the profit that we have set is arbitrary high, it remains always underachieved, it is never overachieved. So we have always $d^+ = 0$ and our target must be to minimized the underachieved amount d^-.

The goal programming formulation of the problem is given by

Minimize $z = d$ ..... (4.4)

subject to constraints $30x_1 + 15x_2 \leq 60$ ..... (4.5)

$20x_1 + 50x_2 \leq 100$ ..... (4.6)

$55x_1 + 45x_2 + d^- - d^+ = 1500$ ..... (4.7)

$x_1, x_2, d^-, d^+ \geq 0$ ..... (4.8)

Where x_1, x_2 are decision variables and d^+, d^- are the deriational variables.

Here the deration from the profit goal 1500 is $d^- = 1500 - (55x_1 + 45x_2)$ As our intention is to maximize the profit $55x_1 + 45x_2$, we must try to minimize the deviation d^-.

The key difference between the LP model (4.1) through (4.3) and the GP model (4.4) through (4.8) is the objective function. In the G.P. model the deviational variable d^- has been used. Also the equation (4.7) is added here as an extra equation which is formed with the help of the objective function $z = 55x_1 + 45x_2$ of the LP model. This equation (4.7) is called the goal equation for the profit goal. Here the targeted goal Rs 1500 is much high and so our objective in GP model is to minimize the under-achievement of the profit goal, and since under-achievement is undesirable, we would like to drive d^- as close to zero as possible. We note here that if the decision maker were interested in achieving the profit goal exactly, then both deviational variables would be included in the objective

function. But in this case the targeted profit goal can not be taken arbitrary high Rs 1500 as we have taken here.

Since the deviational variables can be treated like decision variables, we can solve the GP formulation (4.4) - (4.8) with the traditional simplex method and thereby obtain the final solution as

$$x_1 = \frac{5}{4},\ x_2 = \frac{3}{2},\ d_1^- = 0 \text{ and } d_1^- = 1363\frac{1}{4}$$

So the maximum value of the objective function of the LP model is given from (4.7) as

$$55x_1 + 45x_2 = 1500 - d^- + d^+ = 1500 - \frac{5455}{4} = \frac{545}{4} = 136\frac{1}{4}$$

Thus we get the same solution of the LP model. It appears that little is gained from this G.P. model in comparison with the LP model. But the advantage of GP model will become obvious in future when we will study more complex problems consisting multiple goals with priorities.

108.5 Goal Programming Problem and LPP having no Feasible Solution

In a LPP optimal solution must satisfy all the constraints strictly. So, if there be no point satisfying all the constraints, we say the LPP has no feasible solution. An LPP, in general, is an oversimplified form of reality. So in many situations it may happen that the real problem has a solution but due to over simplification the corresponding LPP gives no feasible solution as answer. In goal programming problem full satisfaction of all constraints is not essential, here optimization means "as close as possible to the indicated goals". So, if a LPP has no feasible solution it may not reflect the actual situation always. In such a situation goal programming technique should be applied to the problem to get the actual answer to the problem.

108.5.1 Example

Let us consider the following LPP arising from a real life problem of a factory involving overtime operation, utilization time of machines, minimum requirement of quality, fulfillment of promised order and profit maximization. The mathematical formulation of this LPP is

Maximize $z = 2x_1 + x_2$

subject to the constraints $3x_1 + 2x_2 \le 12$

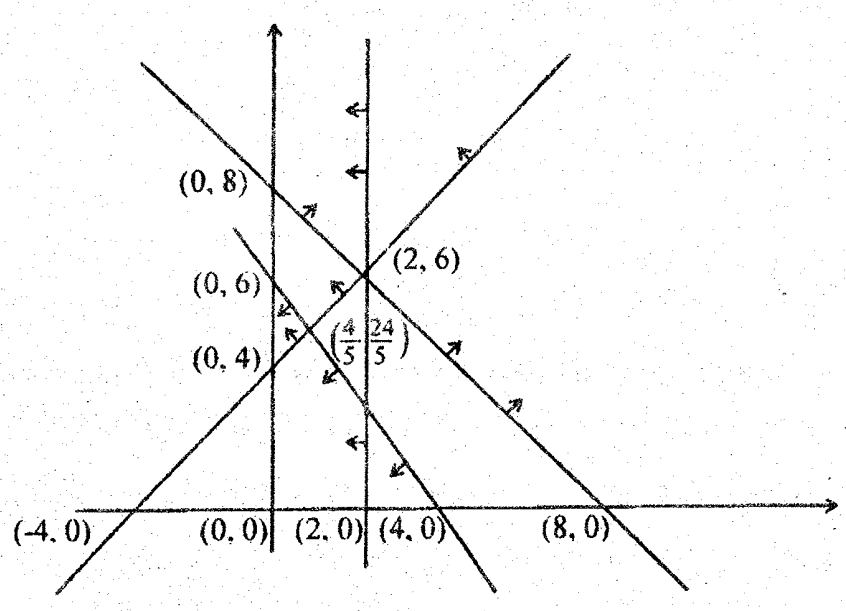
$$x_1 + x_2 \geq 8$$

$$-x_1 + x_2 \geq 4$$

$$x_1 \leq 2$$

and $x_1, x_2 \geq 0$

The graphical solution of this LPP is given below.



From the graph we see that the LPP has no feasible solution.

Considering now this LPP as goal programming problem we consider five management goals instead of the only goal profit maximization. Let us give equal priority to all five goals viz. overtime, time of machines, quality of the product, fulfillment of promised order and profit.

It profit be fixed to the arbitrarily high value Rs 1000, then the objective function z may also be treated as a management goal to acquire the specified profit. Thus, there shall be five management goals in the problem and the real objective may be changed from profit maximization to goal attainment. We note that all the goals may not be attainable. The objective of management may be rather a general approach of attaining the goals as closely as possible. However the equivalent programming problem thus becomes.

Minimize $z = d_1^+ + d_2^- + d_3^- + d_4^+ + d_5^-$

subject to constraints $3x_1 + 2x_2 + d_1^- - d_1^+ = 12$

$x_1 + x_2 + d_2^- - d_2^+ = 8$

$-x_1 + x_2 + d_3^- - d_3^+ = 4$

$x_1 + d_4^- - d_4^+ = 2$

$2x_1 + x_2 + d_5^- - d_5^+ = 1000$

$x_1, x_2, d_i^+, d_i^- \geq 0,\ i = 1, 2, 3, 4, 5$

This goal programming problem will give a compromise solution which is as close as possible to all the five goals.

Important Note :

1. If there is profit maximization goal, then we are to set an arbitrary high goal which can never be achieved. In this case we are to minimize the under-achieved deration d^- from the arbitrary high goal.

2. If there is cost minimization goal, then we are to set an arbitrary low goal which is never desirable to achieve. In this case we are to minimize the over-achieved deviation d^+ from the arbitrary low goal.

   Here for maximization of cx we have $cx + d_1^- - d_1^+ =$ arbitrary high goal g.

   As $cx<g$ we see $d_1^- > 0 \therefore d_1^+ = 0$. So, cx will be maximum when d_1^- is minimum.

   For minimization of cx we have $cx + d_1^- - d_1^+ =$ arbitrary low goal g

   As $cx>g$ we see $d_1^+ > 0 \therefore d_1^- = 0$ Thus $cx - d_1^+ = g$

   So, cx will be minimum when d_1^+ is minimum.

3. For constraints of the type $a_i x \leq b_i$ we have

   $a_i x + d_i^- - d_i^+ = b_i$ In GPP the solution remains as close as possible to the goal. So, $a_i x \leq b_i$ is not satisfied by the solution. We try to minimize the violation of $a_i x \leq b_i$. Hence we try to minimize d_i^+. Similarly, we are to minimize d_i^- if the constraint is $a_i x \geq b_i$.

108.6 G.P. with multiple goals and priorities

The goal programming problems in the examples 108.4.1 and 108.5.1 can be solved by simplex method as the objective function is just linear combination of the deviational variables. But in general the objective function involves priority coefficients which are not real numbers. This is not a linear function and so simplex method can not be applied to such GP problem.

Now we state some GP problems involving multiple goals with priorities. We will show that with change of priorities the answer to the GP will change.

108.6.1 Example

Let a company produces two products A and B. Product A requires 20 hrs. in machine 1 and 10 hrs. in machine 2 whereas product B requires 10 hrs. in machine 1 and 10 hrs. in machine 2. Production time is limited in machine 1 to 60 hrs. and in machine 2 to 40 hrs. Contribution to profits for the two products are respectively Rs 40 and Rs 80. Management's objectives are to maximize the profit and to produce at least two units of each type product. The profit goal and production goal are considered to have equal priorities.

Formulate the goal programming problem.

Solution :

Let x_1 = number of units of product A to produce

x_2 = number of units of product B to produce

Here one goal is to maximize the profit, so we set an arbitrary high goal Rs 1000. Let the durational variables associated with profit goals be d_1^- and d_1^+, deviational variables associated with product A be d_2^- and d_2^+ and those associated with product B to d_3^- and d_3^+ Since all the three goals i.e. profit goal, production of product A goal and production of product B goal are here of equal priority, the objective function is given by $z = d_1^- + d_2^- + d_3^-$.

∴ The goal programming formulation is

Minimize $z = d_1^- + d_2^- + d_3^-$

subject to constraints

$$\begin{aligned} 20x_1 + \quad & 10x_2 & & & \leq 60 \\ 10x_1 + \quad & 10x_2 & & & \leq 40 \\ 40x_1 + \quad & 80x_2 + & d_1^- - & d_1^+ & = 1000 \\ x_1 \quad & & +d_2^- - & d_2^+ & = 2 \\ & x_2 + & d_3^- - & d_3^+ & = 2 \end{aligned}$$

$$x_1,\ x_2,\ d_i^-,\ d_i^+ \geq 0;\ i = 1,2,3$$

The solution to this GPP is found as

$x_1 = 0,\ x_2 = 4,\ d_1^- = 680,\ d_1^+ = 0, d_2^- = 2,\ d_2^+ = 0,\ d_3^- = 0$ and $d_3^+ = 0$

$\therefore$ The profit = Rs(40×0+80×4)=Rs 320

Number of product A produced = 0

Number of product B produced = 2.

The assumed profit goal Rs 1000 is not achieved, the under-achievement is Rs 680. Goal 2 of the production of at least two units of product A is not also fulfilled here. However, goal 3 of the production of at least two units of product B has been achieved.

In example 108.6.1 there were three goals and all of them were equally important i.e. their priorities were equal. No preference of ordering was there in the goals. Question arises, can we provide preference ordering to the goals? Will it affect the solutions? Now, we give answer to such questions.

Preference ordering to the goals is done by using priority coefficients P's. All goals that have top or first priority are assigned an objective function value of P_1, the goals considered to be second in priority are assigned a P_2 value. This process is continued until all goals have been ranked. Here we note that P_1, P_2, ... are not parameters or variables. They do not assume any numerical value. They simply represents levels of priorities. Let us see what happens if we provide priorities in the goals of the example 108.6.1.

108.6.2 Example

In the goal programming problem stated in the example 108.6.1 let the management has established the following goal priorities :

Priority 1 : To meet production goals of two units for each product

Priority 2 : To maximize the profit

Formulate the goal programming problem.

Solution :

Here production goals are given top priority. So we use coefficient P_1 for both deviational variables ..... and .... As maximization of profit has second priority we use the coefficient P_2 with d_1^-.

Thus the revised model becomes

Minimize $z = P_1 d_2^- + P_1 d_3^- + P_2 d_1^-$

subject to constraints

$$\begin{aligned}
2x_1 + \quad & 10x_2 & & & \le 60 \\
10x_1 + \quad & 10x_2 & & & \le 40 \\
40x_1 + \quad & 80x_2 + & d_1^- - & d_1^+ & = 1000 \\
x_1 \quad & & +d_2^- - & d_2^+ & = 2 \\
& x_2 + & d_3^- - & d_3^+ & = 2
\end{aligned}$$

$$x_1, x_2, d_i^-, d_i^+ \ge 0;\ i = 1,2,3$$

The solution to this goal programming problem is

$x_1 = 2,\ x_2 = 2,\ d_1^- = 760,\ d_1^+ = 0, d_2^- = 0,\ d_2^+ = 0,\ d_3^- = 0, d_3^+ = 0$

Here the first priority goal i.e. the production goal is fully achieved. The profit = Rs(40×2+80×2)=Rs 240. So, the amount of profit has come down from Rs 320 to Rs 240 as profit goal is considered in second priority.

We note here that though simplex method can be applied to get the solution of the GPP 108.6.1 but simplex method is not applicable to get the solution of GPP 108.6.2 as the objective function is not a linear expression here. To solve this GPP we are to use modified GP simplex algorithm or graphical method.

Now we study the situation in which the priorities of the two goals in GPP 108.6.2 are interchanged. Naturally, first priority profit goal must provide emphasis on the profit value in the solution here. Let us see.

108.6.3 Example

In example 108.6.2 the goals are interchanged i.e. we have the following priorities

Priority 1 : To maximize the profit

Priority 2 : To meet production goals of two units for each product.

Formulate the goal programming problem.

Solution :

Here only the objective function is changed & the GPP becomes

Minimize $z = P_1 d_1^- + P_2 d_2^- + P_3 d_3^-$

subject to constraints

$$\begin{array}{llllll} 2x_1 + & 10x_2 & & & \leq 60 \\ 10x_1 + & 10x_2 & & & \leq 40 \\ 40x_1 + & 80x_2 + & d_1^- + & d_1^+ & =1000 \\ x_1 & & +d_2^- + & d_2^+ & = 2 \\ & x_2 + & d_3^- + & d_3^+ & = 2 \end{array}$$

$$x_1, x_2, d_i^-, d_i^+ \geq 0;\ i = 1,2,3$$

The solution to this GPP is found as

$x_1 = 0,\ x_2 = 4,\ d_1^- = 680,\ d_1^+ = 0, d_2^- = 2,\ d_2^+ = 0,\ d_3^- = 0, d_3^+ = 0$

Here the profit = Rs(40×0+80×4)=Rs 320.

As expected the profit has increased from Rs 240 to Rs 320. This is because profit goal has given priority 1 here. However, the production goal is not fulfilled completely as it is in second priority.

Now we consider goal programming problem with relative weight of the deviational variables. Sometimes we may wish to give some of the equally ranked goals more importance than others. In such a situation, a differential weight is used to reflect the difference of importance within the same priority level.

108.6.4 Example

In the example 108.6.2 instead of having a goal of producing 2 units of each type of product, we

will set a goal of producing a minimum of 4 units of product A and 6 units of product B. Since product B contributes twice as much to profits as product A, we should produce product B before producing product A. Overtime will be required in producing either quantity of the specified products, as all production capacity are used in producing 2 units of each product. We assume that 50 hrs of overtime is available. We suppose the following priorities for goal attainment.

Priority 1 : Limit total overtime in the two production operations to 50 hrs.

Priority 2 : Meet minimum production goals of 4 units of product A and 6 units of product B. Use differential weights of 1 and 2 respectively, since these reflects the "weighted contribution to profits" of Rs 40 and Rs 80 respectively.

Priority 3 : Maximize the profit.

Solution :

In example 108.6.2 the production time is limited to machine 2 to 40 hrs. To produce 2 units of product A and 2 units of product B, the machine 2 needs (2×10+2×10) hrs i.e. 40 hrs. So all the production capacity of machine 2 have been used. Hence overtime will be required in producing 4 units of product A and 6 units of product B. Thus it will be necessary to add durations variables to constrains $20x_1 + 10x_2 \le 60$ and $10x_1 + 10x_2 \le 40$. We must also add a goal constraint to reflect the goal of limiting overtime to 50 hrs. The modified model of the example 108.6.2 thus becomes

Minimize $$z = P_1 d_6^- + \left(1P_2 d_2^- + 2P_2 d_3^-\right) + P_3 d_1^-$$

subject to constraints

$$\begin{aligned}
20x_1 + \quad & 10x_2 \quad + d_4^- \quad - d_4^+ \quad = 60 \\
10x_1 + \quad & 10x_2 \quad + d_5^- \quad - d_5^+ \quad = 40 \\
40x_1 + \quad & 80x_2 + \quad d_1^- \quad - d_1^+ \quad = 1000 \\
x_1 \quad & \quad + d_2^- \quad - d_2^+ \quad = 4 \\
& x_2 + \quad d_3^- \quad - d_3^+ \quad = 6 \\
d_4^+ + \quad & d_5^+ + \quad d_6^- \quad - d_6^+ \quad = 50
\end{aligned}$$

$$x_1, x_2, d_i^-, d_i^+ \ge 0;\ i = 1, 2, 3, 4, 5, 6$$

Here d_4^+ and d_5^+ represents overtimes. The deviational variable d_6^- reflects the extent to which the 50 hrs. of total overtime is under-utilized. The variable d_6^+ is included in overtime constraint to reflect the possibility of exceeding the allowed overtime 50 hrs.

The solution of this GPP is found to be $x_1 = 1$, $x_2 = 6$, $d_1^- = 480$, $d_1^+ = 0$, $d_2^- = 3$, $d_2^+ = 0$, $d_3^- = 0$, $d_3^+ = 0$, $d_4^- = 0$, $d_4^+ = 20$ $d_5^- = 0$, $d_5^+ = 30$, $d_6^- = 0$, $d_6^+ = 0$ and $Z = 3P_1 + 480P_3$. We note that goal 1 in priority P_1 is achieved i.e. the overtime of 50 hrs. been used fully. The values $d_4^+ = 20$ and $d_5^+ = 30$ shows that 20 hrs. is used in machine 1 and 30 hrs. is used in machine 2.

Goal 2 in priority P_2 is achieved partially. Six units of product B were produced since this portion of the goal had the largest differential weight. But only one unit of product A was produced because of the limited overtime.

Goal 3 in priority P_3 was not achieved here. The total contribution to profit is Rs (40×1+80×6) i.e. Rs 520. The goal is underachieved by d_1^- = Rs 420. In example 108.6.2 the profit was Rs 240, here the profit has increased to the amount Rs 480. The additional profit is achieved due to overtime. But the production goal cannot be met since overtime is limited to only 50 hrs. Because of the differential weights used on the production goals, all 6 units of product B were produced before producing any units of product A.

108.6.5 Example

In example 108.6.4 the weights assigned to priority 2 was $1P_2d_2^- + 2P_2d_3^-$. If we reverse these weights i.e. if the weights be $2P_2d_2^- + 1P_2d_3^-$ then the solution must change. This example will show the importance of assigning weights in the same priority level. The example 108.6.4 is changed to the following.

Minimize $$z = P_1d_6^- + \left(2P_2d_2^- + 1P_2d_3^-\right) + P_3d_1^-$$

subject to constraints

$$\begin{aligned}
20x_1 + \quad & 10x_2 \quad + d_4^- \quad - d_4^+ \quad = 60 \\
10x_1 + \quad & 10x_2 \quad + d_5^- \quad - d_5^+ \quad = 40 \\
40x_1 + \quad & 80x_2 + \quad d_1^- \quad - d_1^+ \quad = 1000 \\
x_1 \quad & \quad + d_2^- \quad - d_2^+ \quad = 4 \\
& x_2 + \quad d_3^- \quad - d_3^+ \quad = 6 \\
d_4^+ + \quad & d_5^+ + \quad d_6^- \quad - d_6^+ \quad = 50
\end{aligned}$$

$$x_1, x_2, d_i^-, d_i^+ \geq 0;\ i = 1, 2, \ldots\ldots, 6$$

The solution of the example 108.6.4 is changed here to $x_1 = 4$, $x_2 = 1.5$, $d_1^- = 720$, $d_1^+ = 0$, $d_2^- = 0$,

$d_2^+ = 0,\ d_3^- = 4.5,\ d_3^+ = 0,\ d_4^- = 0,\ d_4^+ = 35,\ d_5^- = 0,\ d_5^+ = 15,\ d_6^- = 0,\ d_6^+ = 0$ and $Z = 4.5P_2 + 720P_3$. We see that because of the large weight i.e. 2 assigned to the product A we get 4 units of product A instead of 1. The less weight 1 assigned to product B gives only 1.5 unit of product B instead of 6. This shows the importance of differential weights assigned to the same priority level.

Now we show the importance of the ranking (ordering) of priorities. Let us reverse the first two goals in example 108.6.4 and see its effect on the solution. For this we consider the following example.

108.6.6 Example

In the example 108.6.4 we change the priorities as follows

Priority 1 : Meet minimum production goals of 4 units of product A and 6 units of product B. Use differential weights of 1 and 2 respectively to profits Rs 40 and Rs 80.

Priority 2 : Limit total overtime in the two production operations to 50 hrs.

Priority 3 : Maximize the profit.

Solution :

Here in goal 1 i.e. in priority P_1, 4 units of product A and 6 units of product B are to be produced.

In goal 2 i.e. in priority P_2, total overtime in the two production operations are limited to 50 hrs.

Hence the new GPP become

Minimize $z = 1P_1d_2^- + 2P_1d_3^- + P_2d_6^- + P_3d_1^-$

subject to constraints

$$\begin{aligned}
20x_1 + 10x_2 + d_4^- - d_4^+ &= 60\\
10x_1 + 10x_2 + d_5^- - d_5^+ &= 40\\
40x_1 + 80x_2 + d_1^- - d_1^+ &= 1000\\
x_1 + d_2^- - d_2^+ &= 4\\
x_2 + d_3^- - d_3^+ &= 6\\
d_4^+ + d_5^+ + d_6^- - d_6^+ &= 50
\end{aligned}$$

$$x_1, x_2, d_i^-, d_i^+ \geq 0;\ i = 1, 2, 3, 4, 5, 6$$

The solution to this problem is $x_1 = 4,\ x_2 = 6,\ d_1^- = 360,\ d_1^+ = 0,\ d_2^- = 0,$

$d_2^+ = 0,\ d_3^- = 0,\ d_3^+ = 0,\ d_4^- = 0,\ d_4^+ = 80,\ d_5^- = 0,\ d_5^+ = 60,\ d_6^- = 0,\ d_6^+ = 90$ and $Z = 90P_2 + 360P_3$.

Comparing these results with those of the example 108.6.5 it is seen that because the production goals have top priority, both production levels i.e. x_1=4, x_2=6 are met. But this occurs at a cost of an additional 90 hrs. of overtime as $d_6^+ = 90$ This means that 140 overtime hours are required in order to meet the production goals.

108.7 Graphical Solution of Goal Programming Problems

If the GP problem contains only two decision variables then graphical approach can be used to solve GP problem.

The following are the steps to be followed in this approach.

Step 1. Graph or plot all structural constraints and identify the feasible region. It no structural constraints exist, take the feasible region as the first quadrant i.e. $x_1 \geq 0$, $x_2 \geq 0$

Step 2. Graph the lines corresponding to the goal constraints. This is accomplished by setting the deviational variables in the goal constraint to zero and plotting the resulting equation.

Step 3. Identify the top-priority solution. This is accomplished by determining the point or points within the feasible region that satisfy the highest priority goal.

Step 4. Move to the goal having the next highest priority and determine the 'best' solution, such that this 'best' solution does not degrade the solutions already achieved for goals of higher priority.

Step 5. Repeat step 4 until all priority levels have been investigated.

To discuss the graphical procedure, we consider the following example.

108.7.1 Example

A company manufactures two types of product using two departmental operations. To produce one unit of product A department 1 requires 2 hrs. of production time and department 2 requires 3 hrs. of production time. On the other hand for one unit of product B department 1 and 2 requires respectively 4 hrs. and 3 hrs. At present 80 hrs. of labour are available each week in each of the departments. This labour time is a somewhat restrictive factor since the company has a general policy of avoiding overtime. The profit per unit of product A and B are respectively Rs 30 and Rs 40. Management has set the following goals.

Priority 1. Avoid overtime operations in each department.

Priority 2. Prior sales records indicate that, on the average, a minimum 10 product A and 10 product B can be sold weakly. Management like to fulfil these sales goals. Since production time may limit producing these number of each product, and since product B has a higher profit margin, the sale goals should be weighted by the profit contributions for respective product.

Priority 3. Maximize the profit.

Solution :

The GP model for the problem is given by

Minimize $z = P_1 d_1^+ + P_1 d_2^+ + 30P_2 d_3^- + 40P_2 d_4^- + P_3 d_5^-$

subject to constraints

$$\begin{array}{rrrrr} 2x_1 + & 4x_2 & +d_1^- & -d_1^+ & =80 \\ 3x_1 + & 3x_2 & +d_2^- & -d_2^+ & =80 \\ x_1 + & & d_3^- & -d_3^+ & =10 \\ & x_2 & +d_4^- & -d_4^+ & =10 \\ 30x_1 + & 40x_2 + & d_5^- & -d_3^+ & =1440 \end{array}$$

$$x_1, x_2, d_i^-, d_i^+ \geq 0;\ i = 1, 2, 3, 4, 5.$$

The value 1440 for the right hand side is an arbitrary high profit goal.

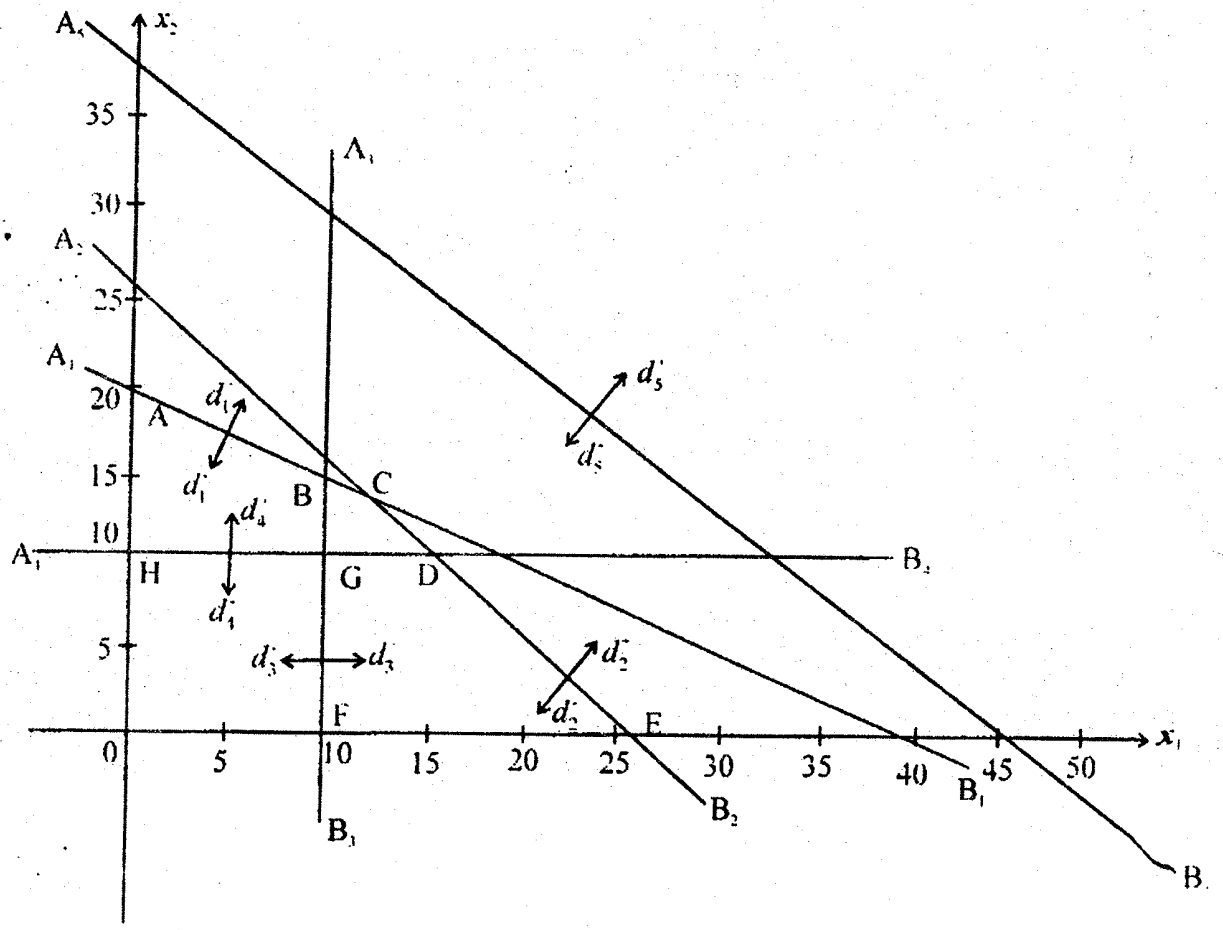
For better understanding of how goal programming procedure works we solve this problem in detail using different steps of graphical method.

Step 1. Graph all structural constraints : Since there is no structural constraint, the feasible region is the first quadrant i.e. the area given by $x_1 \geq 0$ and $x_2 \geq 0$.

Step 2. Graph all goal constraints : By setting the deviational variables to zero in each goal constraint, we plot the resulting goal equations. In the figure these lines are A_1B_1, A_2B_2, A_3B_3, A_4B_4 and A_5B_5.

The effect of a deviational variable on a goal equation is to shift the lines corresponding to the equation so that the new line is parallel to the original line. The value of the deviational variable reflects the degree of shift. The directed arrows from the goal line reflects the impact of the deviational variables on the equation.

Step 3. Identify the top-priority solution : Since d_1^+ and d_2^+ have a priority coefficient P_1 in the objective function, we are to consider the goal constraints 1 and 2 first. Here the objective is to minimize d_1^+ and d_2^+. If we set d_1^+ to zero, the area of feasibility is the area below the line A_1B_1. If we set d_2^+ to zero the area of feasibility is the area below the line A_2B_2. The feasible area common to both is that below the line AC and CE. Any point in this area will satisfy $d_1^+ = 0$ and $d_2^+ = 0$.



We note here on important point that the entire area OACE is the optimal solution of the top priority. It is wrong to conclude that the vertex C is the optimal solution of the top priority.

Step 4. Move to next highest priority and determine 'best' solution, without violating prior goals : The deviational variables d_3^- and d_4^- have a priority coefficient P_2 in the objective

function. Therefore we are to examine these goal constraints next. Here $P_2d_4^-$ has differential weight 40 whereas $P_2d_3^-$ has differential weight 30. Hence we should try to drive d_4^- to zero before considering d_3^- $(as\ 40>30)$. Now d_4^- is zero in the region above A_4B_4. So within the optimal solution of top priority i.e. within OACEO, $d_4^- = 0$ gives the area AHDCA. Again on the right of the line A_3B_3 we have $d_3^- = 0$. So within AHDCA we have $d_3^- = 0$ in the region BGDCB. Thus any point in the area BGDCB satisfy the conditions $d_1^+ = 0$, $d_2^+ = 0$, $d_4^- = 0$ and $d_3^- = 0$ and hence gives the optimal solution of priority 1 and 2.

Step 5. Repeat Step 4 until all priorities are investigated. The third and last priority is P_3 and the deviational variable associated with it is d_5^-. We have been able to drive d_1^+, d_2^+, d_4^-, d_3^- to zero which means that we have been able to meet the goals in P_1 and P_2 fully. If we can drive d_5^- to zero, then all the goals will be achieved. But here $d_5^- = 0$ above the line A_5B_5 which lies outside the obtained solution region BGDCB. So $d_5^- > 0$ within BGDCB, which means P_3 level goal cannot be achieved fully, it can only be achieved partially with some positive value of d_5^-. In other words, in order to have an acceptable solution that does not deteriorate the achievement of goals with a higher priority, d_5^- must be positive. Obviously, we must try to have the value of d_5^- as small as possible.

Therefore, to identify the solution(s) that best satisfy our objective of minimizing d_5^- we draw lines parallel to A_5B_5 till we have a contact with the solution region of higher priorities i.e. the region BGDCB. This is the point C which is the point of intersection of the lines A_1B_1 and A_2B_2 and so has the coordinates (13.33, 13.33). Hence the solution is $x_1 = 13.33$, $x_2 = 13.33$. From the goal constraints we can find the final solution as $x_1 = 13.33$, $x_2 = 13.33, d_1^- = 0, d_1^+ = 0, d_2^- = 0, d_2^+ = 0, d_3^- = 0, d_3^+ = 3.33$ $d_4^- = 0, d_4^+ = 3.33, d_5^- = 506.9$ and $d_5^+ = 0$. Here the minimum value of d_5^- is 506.9 and so the goal in P_3 is not achieved fully. Here exactly 80 hrs will be used in each departmental operation and the profit will be Rs(30×13.33+40×13.33) i.e. Rs 9333.33.

108.8 Illustrative Examples

We shall use graphical procedure to solve the following examples.

108.8.1 Examples

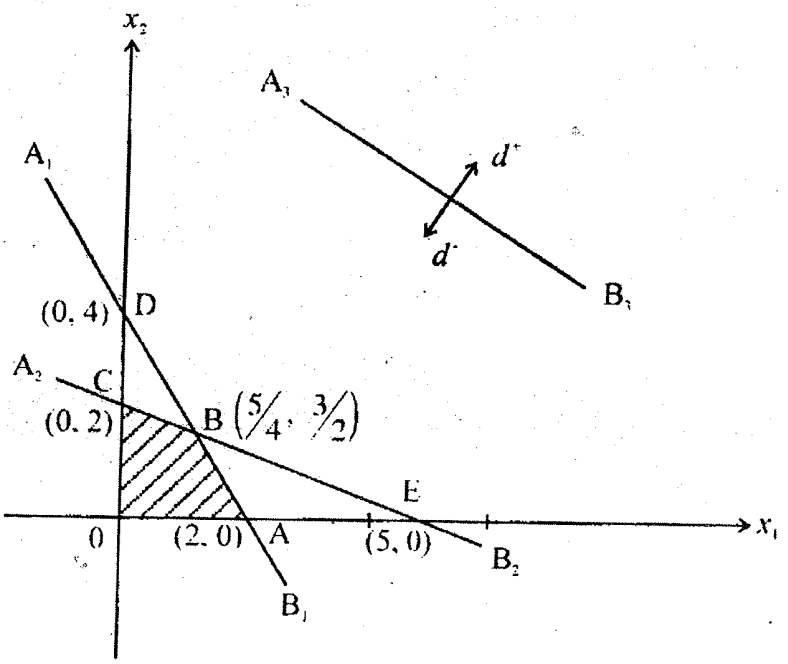
Minimize $z = d^-$

subject to constraints

$$30x_1 + 15x_2 \le 60$$
$$20x_1 + 50x_2 \le 100$$
$$55x_1 + 45x_2 + d^- - d^+ = 1500$$
$$x_1, x_2, d^-, d^+ \ge 0$$

Solution :

This example is considered in the article 108.4 and is the goal programming model of the LPP (4.1) - (4.3). However, here we solve this GPP using graphical method. We draw the structural constraints and proceed accordingly.



Here the arbitrary high goal assigned to the profit is Rs 1500. The structural constraints gives the feasible region OABCO. The goal constraint $55x_1 + 45x_2 = 1500$ is the line A_3B_3. Here d^- is positive in the region below the line A_3B_3. In order to have the optimal solution we are to draw lines parallel to A_3B_3 till we have a contact with the feasible region OABCO. The first contact point found is $B\left(\frac{5}{4}, \frac{3}{2}\right)$.

Hence the solution is $x_1 = 5/4$, $x_1 = 3/2$, $d^+ = 0$. The goal equation gives $d^- = 1363\frac{1}{4}$. The maximum profit is Rs. 136¼.

108.8.2 Example

This example was considered in the article 108.6.1 the example is

Minimize $z = d_1^- + d_2^- + d_3^-$

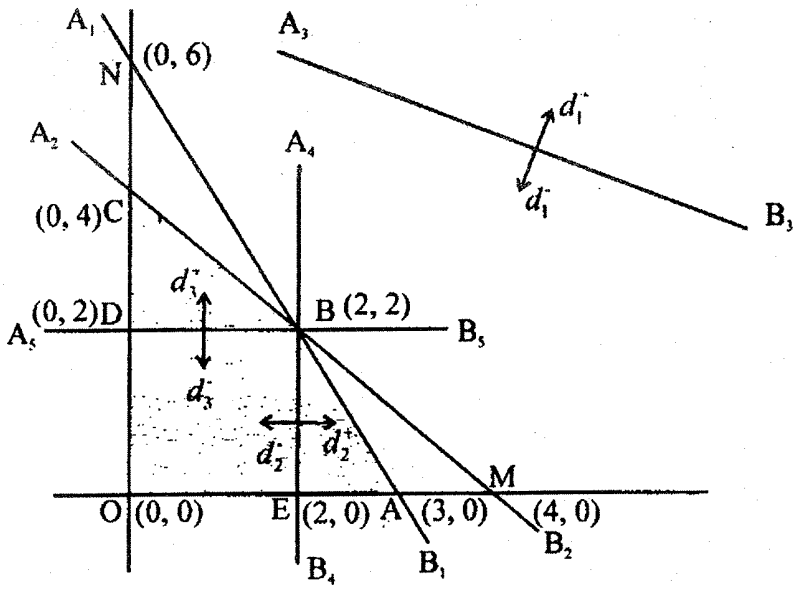
subject to constraints

$$\begin{array}{llllll} 20x_1 + & 10x_2 & & & \le 60 \\ 10x_1 + & 10x_2 & & & \le 40 \\ 40x_1 + & 80x_2 + & d_1^- & -d_1^+ & =1000 \\ x_1 & & +d_2^- & -d_2^+ & =2 \\ & x_2 + & d_3^- & -d_3^+ & =2 \end{array}$$

$$x_1, x_2, d_i^-, d_i^+ \ge 0;\ i = 1, 2, 3$$

Solution :

Here all goals have equal priority. The graph shows the structural constrains A_1B_1, A_2B_2 and goal constraints A_3B_3, A_4B_4 and A_5B_5. Also directed arrows are shown from the goal lines to show their positive values.



The structural constraints gives the feasible region as OABCO. From the goal constraints we have the following values of the deviational variables at different verities of the feasible region.

At $O(0,0): x_1 = 0,\ x_2 = 0,\ d_1^+ = 0,\ d_1^- = 1000,\ d_2^+ = 0,\ d_2^- = 2,\ d_3^+ = 0,\ d_3^- = 2$

At $A(3,0): x_1 = 3,\ x_2 = 0,\ d_1^+ = 0,\ d_1^- = 880,\ d_2^+ = 1,\ d_2^- = 0,\ d_3^+ = 0,\ d_3^- = 2$

At $B(2,2): x_1 = 2,\ x_2 = 2,\ d_1^+ = 0,\ d_1^- = 760,\ d_2^+ = 0,\ d_2^- = 0,\ d_3^+ = 0,\ d_3^- = 0$

At $C(0,4): x_1 = 0,\ x_2 = 4,\ d_1^+ = 0,\ d_1^- = 680,\ d_2^+ = 0,\ d_2^- = 2,\ d_3^+ = 2,\ d_3^- = 0$

Here the objective is to minimize $z = d_1^- + d_2^- + d_3^-$

At $O: z = 1000 + 2 + 2 = 1004$

At $A: z = 800 + 0 + 2 = 882$

At $B: z = 760 + 0 + 0 = 760$

At $C: z = 680 + 2 + 0 = 682$

$\therefore$ z attains its minimum value at $C(0,4)$ and is 682. Hence the solution of the GPP is

$x_1 = 0,\ x_2 = 4,\ d_1^+ = 0,\ d_1^- = 680,\ d_2^+ = 0,\ d_2^- = 2,\ d_3^+ = 2$ and $d_3^- = 0$ the profit = Rs(40×0+80×4)=Rs 320.

108.8.3 Example

Now we solve the example 108.6.6 graphically. The GPP is

Minimize $z = \left(1P_1 d_2^- + 2P_1 d_3^-\right) + P_2 d_6^+ + P_3 d_1^-$

subject to

$$\begin{array}{rrrrr}
20x_1 & +10x_2 & +d_4^- & -d_4^+ & =60 \\
10x_1 & +10x_2 & +d_5^- & -d_5^+ & =40 \\
40x_1 & +80x_2 & +d_1^- & -d_1^- & =1000 \\
x_1 & & +d_2^- & -d_2^- & =4 \\
 & x_2 & +d_3^- & -d_3^- & =6 \\
d_4^+ & +d_5^+ & +d_6^- & -d_6^+ & =50
\end{array}$$

$x_1, x_2, d_i^-, d_i^+ \geq 0;\ i = 1, 2, \ldots, 6.$

Solution :

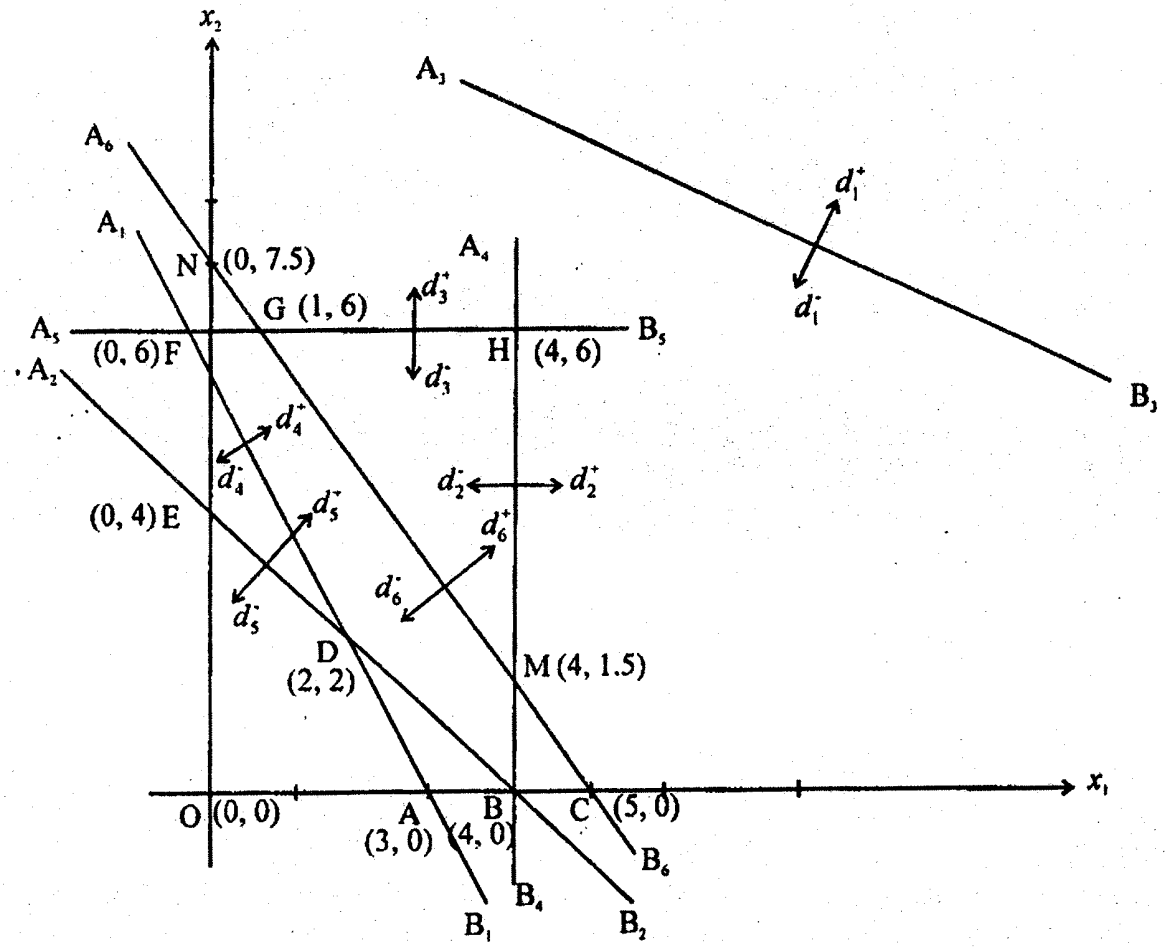
Noting the statement of this example in the article 108.6.6, here d_4^+ and d_5^+ represent, respectively, the overtime required in operations 1 and 2. The deviational variable d_6^- reflects the extent to which the 50 hrs of total overtime is under-utilized and the deviational variable d_6^+ reflects the possibility of exceeding the 50 hrs.

To draw the line related to overtime goal constraint $d_4^+ + d_5^+ + d_6^- - d_6^+ = 50$ we replace d_4^+ and d_5^+ in terns of x_1 and x_2 using the first two goal constraint and obtain

$$20x_1 + 10x_2 - 60 + 10x_1 + 10x_2 - 40 + d_6^- - d_6^+ = 50$$

i.e., $30x_1 + 20x_2 + d_6^- - d_6^+ = 150$

We now draw the goal lines related to all goals. These lines are respectively A_1B_1, A_2B_2, ..., A_6B_6.



The top priority is $1P_1d_2^- + 2P_1d_3^-$. Here $P_1d_3^-$ has more weight. Hence we have to try to minimize d_3^- first. Now $d_3^- = 0$ on or above the line A_5B_5. So the feasible area for $P_1d_3^-$ is the region on and above the line FB_5. In this region we are to minimize $P_1d_2^-$. Now $d_2^- = 0$ on the right of the line A_4B_4. Hence considering P_1 we get the feasible region as A_4HB_5.

The second priority goal is $P_2d_6^+$ and $d_6^+ = 0$ on and below the line A_6B_6. So within the feasible region A_4HB_5 the value of d_6^+ is positive. Minimum d_6^+ occurs at the point $H(4,6)$.

Hence the 'best' solution obtained considering P_1 & P_2 is $x_1 = 4$, $x_2 = 6$. For the third priority, P_3, we have $d_1^- > 0$ at $H(4,6)$ and we have to accept it.

Thus the accepted final solution to the GPP is the point $H(4,6)$ with the values of the variables as $x_1 = 4$, $x_2 = 6$, $d_1^- = 360$, $d_1^+ = 0$, $d_2^- = 0$, $d_2^+ = 0$, $d_3^- = 0$, $d_3^+ = 0$, $d_4^- = 0$, $d_4^+ = 80$, $d_5^- = 0$, $d_5^+ = 60$,

$d_6^- = 0$, $d_6^+ = 90$

Hence $z = 90P_2 + 360P_3$

Here P_1 is fully satisfied at the cost of 90 hrs overtime.

108.8.4 Example

In the example 108.8.3 let us change the objective function to

$$z = P_1d_6^+ + (1P_2d_2^- + 2P_2d_3^-) + P_3d_1^-$$

This example was considered in 108.6.4.

Here the feasible region which minimizes $P_1d_6^+$ i.e. $d_6^+ = 0$ is the region on and below the line A_6B_6.

In the second priority we are to minimize d_3^- first and $d_3^- = 0$ in the region on and above the line A_5B_5.

So, we get the feasible region as NFG. Now we consider $P_2d_2^-$. In NFG d_2^- is minimum $G(1,6)$. Hence $G(1,6)$ is the best solution and at G we are to accept some positive value of d_1^- for P_3. Thus the required solution is $x_1 = 1$, $x_2 = 6$, $d_1^- = 480$, $d_1^+ = 0$, $d_2^- = 3$, $d_2^+ = 0$, $d_3^- = 0$, $d_3^+ = 0$, $d_4^- = 0$, $d_4^+ = 20$ $d_5^- = 0$, $d_5^+ = 30$, $d_6^- = 0$, $d_6^+ = 0$ and $z = 3P_1 + 480P_3$

108.8.5 Example

In the example 108.8.4, let us interchange the weights in P_2 i.e. let the objective function be

$$z = P_1 d_6^+ + \left(2P_2 d_2^- + 1P_2 d_3^-\right) + P_3 d_1^-$$

This example was considered in 108.6.5.

Considering $P_1 d_6^+$ we get the feasible region ONC. Now, considering $2P_2 d_2^-$ the feasible region reduces to MBC. Within MBC, d_3^- is minimum at $M(4,1.5)$.

Considering $P_3 d_1^-$ we remain at the same point $M(4,1.5)$. Hence the required solution is $M(4,1.5)$.

i.e. $x_1 = 4,\ x_2 = 1.5,\ d_1^- = 720,\ d_1^+ = 0,\ d_2^- = 0,\ d_2^+ = 0,\ d_3^- = 4.5,\ d_3^+ = 0,\ d_4^- = 0,\ d_4^+ = 35,$

$d_5^- = 0,\ d_5^+ = 15,\ d_6^- = 0,\ d_6^+ = 0$ and $z = 45P_1 + 720P_3$

108.9 Summary

The need of GPP is discussed here stating its relation with LPP. Multigoal programming with different priorities is considered in details. Also effect of adding differential weights in the same priority level is taken into consideration. Graphical method of solving solution is explained and used in many examples for illustration.

108.10 Self Assessment Questions

1. Identify the major differences between linear programming and goal programming.
2. Explain the following terms :
   i) Deviational variables
   ii) Differential weights
   iii) Priorities in the goals
3. The production manager faces the problem of job allocation among three of his teams. The processing rate of the three teams are 5, 6 and 8 units per hour respectively. The

normal working hours for each team are 8 hours per day. The production manager has the following goals for the next day in order of priority :

i) The manager wants to avoid any underachievement of production level, which is set at 180 units of product.

ii) Any overtime operation of team 2 beyond 2 hrs and team 3 beyond 3 hrs should be avoided.

iii) Minimize the sum of overtime.

Formulate this as a goal programming problem and solve it.

4. A manufacturing firm produces two types of product A and B. According to past experience, production of either Product A or Product B requires an average of one hour in the plant. The plant has a normal production capacity of 400 hrs a month. The marketing department of the firm reports that because of limited market, the maximum number of Product A and Product B that can be sold in a month are 240 and 300 respectively. The net profit from the sale of Product And Product B are Rs 800 and Rs 400 respectively. The manager of the firm has set the following goals arranged in the order of importance.

P_1 : He wants to avoid any under utilization of normal production capacity.

P_2 : He wants to sell maximum possible units of Product A and Product B. Since the net profit from the sale of product A is twice the amount from that of Product B, the manager has twice as much desire to achieve sales for Product A as for Product B.

P_3 : He wants to minimize the overtime operation of the plant as much as possible.

Formulate this as GPP and solve it graphically.

5. Delton Electronics manufactures two types of TV sets. One TV set, the Deluxe, requires 2 hours in assembly, while the other, the supreme, requires 4 hours assembly time. The normal assembly operation is limited to 80 hours per week. Marketing surveys indicate

that no more than 60 Deluxe and 30 Supreme TV sets should be produced each week. The net profit from the Deluxe model is Rs 100 each and is Rs 150 each from the Supreme model. The company president has siated the following objectives in order of priority.

P_1 : Maximize total profit.

P_2 : Minimize overtime operation.

P_3 : Sell as many TV sets as possible. Since the net profit from the Supreme model is 2 times that from the Deluxe model, the president has 2 times as much desire to maximize the sales of the Supreme model as he does for the Deluxe model.

Formulate the GPP and solve it graphically.

6. A camera Company manufactures two types of 35 mm cameras. The production process for manufacturing the cameras in such that two departmental operations are required. To produce their standard camera requires 4 hrs of production time in department 1 and 3 hrs in department 2. To produce their deluxe model requires 2 hrs of production time in department 1 and 3.5 hrs in department 2. Currently, 180 hrs of labour are available each week in each of the department. This labour time is a somewhat restrictive factor since the company has a general policy of avoiding overtime, if possible. The manufacturer's profits on each standard camera is Rs 60, while the profit on the deluxe model is Rs 80. Management has set the following goals :

P_1 : Avoid overtime operations in each department.

P_2 : Prior sales records indicate that, on the average, a minimum of 10 standard and 12 deluxe cameras can be sold weekly. Management would like to meet these sales goals. Since production time may limit producing this number of each camera, and since the deluxe camera has a higher profit margin, the sales goals should be weighted by the profit contribution for the respective cameras.

P_3 : Maximize profits.

Formulate the GPP and solve it graphically.

108.11 Suggested Further Readings

1. Lee. S. M. and L. J. Moore, Introduction to Decision Science. New York; Petrocellicharter Inc. 1975.

2. Lee S. M. Goal Programming for Decision Analysis. Philadelphia : Auerbach Publishers. 1972.

3. M. P. Gupta and J. K. Sharma. Operations Research for Management. National Publishing House. New Delhi 1984

4. S. D. Sharma. operations Research, Kedar Nath Ram Nath.

5. Kanti Swarup, P. K. Gupta, Man Mohan, Operations Research, Sultan Chand & Sons.

"Learner's Feed-back"

After going through the Modules/ Units please answer the following questionnaire.
Cut the portion and send the same to the Directorate.

To
The Director
Directorate of Distance Education,
Vidyasagar University,
Midnapore- 721 102.

1. The modules are : (give ✓ in appropriate box)
   ☐ Easily understandable; ☐ very hard; ☐ partially understandable.
2. Write the number of the Modules/Units which are very difficult to understand :
   .....
   .....
   .....
3. Write the number of Modules/ Units which according to you should be re-written :
   .....
   .....
   .....
4. Which portion/page is not understandable to you? (Mention the page no. And Portion)
   .....
   .....
   .....
   .....
5. Write a short comment about the study material as a learner.
   .....
   .....
   .....
   .....
   .....

.....
(full Signature of the Leaner)

Enrolment No. .....

Date : .....

Phone/ Mobile No. .....

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