2025

M.Sc. 1st Semester Examination

APPLIED MATHEMATICS

Paper : MTMC401X1

[Measure Theory]

Full Marks : 25 Time : One Hour

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
Notations and symbols have their usual meanings.

Group - A

Answer any *two* questions : 2×2=4

1. Show that the set of all rational numbers is a null subset of $\mathbb{R}$. [CO1]

2. Let X be a measurable space and $\chi_E : X \to \mathbb{R}$ be a measurable function, where

$$\chi_E(x) = \begin{cases} 1 \text{ if } x \in E \\ 0 \text{ if } x \notin E \end{cases}.$$ Is E a measurable set in X?

[CO1]

3. Define σ-algebra with an example. [CO1]

4. Define measure with an example. [CO1]

P.T.O.

Group - B

Answer any *two* questions : 4×2=8

5. Let μ be a measure on a σ-algebra $\mathfrak{M}$. Then show that $\mu(A_n) \to \mu(A)$ as $n \to \infty$ if $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathfrak{M}$ and $A_1 \subset A_2 \subset A_3 \subset \cdots$. [CO1]

6. Suppose $f : X \to [0, \infty]$ is measurable and $\phi(E) = \int_E f \, d\mu$ for every measurable set E in X. Show that ϕ is a measure and $\int g \, d\phi = \int gf \, d\mu$ for every measurable function g on X with range in $[0, \infty]$.

   [CO1]

7. Show that every bounded Riemann integrable function is Lebesgue integrable and the two integrals are equal in this case. [CO2]

8. Construct the Cantor set. Show that it is a null set.

   [CO1]

Group - C

Answer any *one* question : 8×1=8

9. (i) Let $f(x) = \dfrac{1}{x^p}$ if $0 < x \leq 1$ and $f(0) = 0$. Find necessary and sufficient condition on p such that $f \in L^1[0,1]$. Compute $\int_0^1 f(x)\lambda(x)$ in that case.

   [CO2]

(ii) Let $f : X \to [0,\infty]$ be measurable, $E \in \mathfrak{M}$ and $\int_E f\, d\mu = 0$. Show that $f = 0$ a.e. on E.

5+3 [CO2]

10. (i) Let s_1 and s_2 be nonnegative simple measurable functions on a measurable space X. Show that

$$\int_X (s_1 + s_2) d\mu = \int_X s_1\, d\mu + \int_X s_2\, d\mu .$$ [CO2]

(ii) State Fatou's lemma. Give an example to show that strict inequality can occur in Fatou's lemma.

4+4 [CO2]

Internal Assessment : 5 marks
