

2025

M.Sc. 1st Semester Examination

PHYSICS

Paper : PHSC404X0

[Quantum Mechanics-I]

Full Marks : 50

Time : Two Hours

*The figures in the margin indicate full marks.**Candidates are required to give their answers in their own words as far as practicable.*

Group - A

Answer any *four* questions : $2 \times 4 = 8$

1. For the angular momentum operators $\hat{J}_i$, show that

$$[\hat{J}_+, \hat{J}_-] = 2\hbar\hat{J}_z \text{ where } \hat{J}_{\pm} = \hat{J}_x \pm i\hat{J}_y.$$

2. Derive the expression for position space wave function $\langle x | \alpha \rangle = \psi_{\alpha}(x)$ as an integral over the momentum space wave function $\langle p | \alpha \rangle = \phi_{\alpha}(p)$ starting from the state $|\alpha\rangle$.

3. Show that Clebsch-Gordan coefficients vanish unless $m = m_1 + m_2$, where the symbols have their usual meanings.

P.T.O.

4. For a one dimensional harmonic oscillator, using the ladder operators $\hat{a}$ and $\hat{a}^\dagger$, show that the eigenvalues of the number operator $\hat{N} = \hat{a}^\dagger \hat{a}$ are non-negative integers.
5. Show that if a system possesses a symmetry, then it leads to degeneracy of energy eigenvalues.
6. Write down the matrix representation of the spin $-\frac{1}{2}$ operator, $\hat{S}_x$ and find its normalized eigenvectors in the $|\pm, z\rangle$ basis. Further write down the unitary matrix U such that $\hat{U}|+, z\rangle = |+, x\rangle$.

Group - B

Answer any *four* questions : 4×4=16

7. For a one dimensional harmonic oscillator obtain the matrix representations of the operators $\hat{X}$ and $\hat{P}$. Verify that $[\hat{X}, \hat{P}] = i\hbar$.
8. Consider a two-state system with orthonormal basis states $\{|1\rangle, |2\rangle\}$. The Hamiltonian operator for the system is given by $\hat{H} = \epsilon(|1\rangle\langle 1| + |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|)$ where ϵ is a constant having the dimension of energy. Suppose it is known that at time $t = 0$ the system is in state $|\psi(0)\rangle = |1\rangle$. Find $|\psi(t)\rangle$.
9. Given that $\langle x | \hat{P}_x | \alpha \rangle = -i\hbar \frac{\partial}{\partial x} \langle x | \alpha \rangle$, where $|\alpha\rangle$ is any general state. Show that $\langle x | p_x \rangle = (2\pi\hbar)^{-1/2} e^{ip_x x/\hbar}$.

10. Two different spin $\frac{1}{2}$ particles have a Hamiltonian $\hat{H} = \alpha + \beta \hat{S}_1 \cdot \hat{S}_2 + \gamma (\hat{S}_{1z} + \hat{S}_{2z})$ where α, β and γ are constants. Find the energy eigenvalues and the normalized eigenvectors of the system.
11. Find the matrix representation of the angular momentum operator $\hat{J}_y$ for $j=1$ in the $|j, m\rangle$ basis such that the $\hat{J}_z |j, m\rangle = m\hbar |j, m\rangle$.
12. Consider a state $|\psi\rangle$ and a Hamiltonian $\hat{H}$ whose matrix representations are given as

$$|\psi\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad [\hat{H}] = \frac{E_0}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

where E_0 is a constant. If energy is measured, what values would we obtain and with what probabilities?

Group - C

Answer any *two* questions : $8 \times 2 = 16$

13. (a) The Hamiltonian operator for a spin-1/2 system is given by $\hat{H} = \omega \hat{S}_z$ where ω is a real constant. The state of the system at $t = 0$ is $|+, x\rangle$ find the expectation values of the operators $\hat{S}_x, \hat{S}_y, \hat{S}_z$ as a function of time using the Heisenberg picture.

P.T.O.

- (b) Consider a one-dimensional harmonic oscillator with the ground state wave function $\psi_0(x) = e^{-x^2/(2x_0^2)}$, where x_0 is a constant. Find the value of the coefficient α so that $\psi_2(x) = (1 + \alpha x^2)e^{-x^2/(2x_0^2)}$ is orthogonal to $\psi_0(x)$. 5+3

14. If the eigenvalues of $\hat{J}^2$ and $\hat{J}_z$ are given by the eigenvalue equations

$$\hat{J}^2 \left| \lambda, m \right\rangle = \lambda \hbar^2 \left| \lambda, m \right\rangle \text{ and } \hat{J}_z \left| \lambda, m \right\rangle = m \hbar \left| \lambda, m \right\rangle, \text{ then}$$

show that (a) $\lambda \geq m^2$ (b) $\lambda = j(j+1)$ and

(c) $-j \leq m \leq j$, where the symbols have their usual meanings. 3+2+3

15. (a) Consider the angular momentum addition of two spin-half particles. Compute all the associated Clebsch-Gordan coefficients and rewrite all the states in the $|j_1, j_2; j, m\rangle$ basis in terms of those in $|j_1, j_2; m_1, m_2\rangle$ basis. You must show all steps involved in the computation.

- (b) If $|\alpha\rangle$ is an eigenstate of the parity operator $\hat{\pi}$ show that the corresponding eigenfunction $\psi_\alpha(x)$ is either an even or an odd function of x .

6+2

16. (a) Consider a one-dimensional potential that is invariant under lattice translation such that $\hat{\tau}(a)^\dagger V(x) \hat{\tau}(a) = V(x+a) = V(x)$. Show that the state $|\theta\rangle$ defined as $|\theta\rangle = \sum_{n=-\infty}^{\infty} e^{in\theta} |n\rangle$ is an eigenstate of the lattice translation operator $\hat{\tau}(a)$. Further using the tightbinding approximation, find the energy eigenvalues as a function of θ .
- (b) A spin-half system has a Hamiltonian $H = \omega \hat{S}_z$. If the system is in the state $|+, x\rangle$ at time $t = 0$. Find the probabilities of finding the system in the states $|\pm, x\rangle$ as function of time. 4+4

Internal Assessment : 10 marks
