Design and Experiments and Collection of data

Experimentation and making inferences are twin essential features of general scientific methodology. Statistics as a scientific discipline is mainly designed to achieved these objectives. It is generally concerned with problems of inductive inferences in relation to stochastic models describing random phenomena. When faced with the problems of studying a random phenomenon, the scientist, in general, may not have complete knowledge of the true variant of the phenomenon under study. A statistical problem arises when he is interested in the specific behaviour of the unknown variant of the phenomenon.

After a statistical problem has been set-up, the next step is to perform experiments for collecting information on the basis of which inferences can be made in the best possible manner.

The methodology for making inferences has three main aspects. First, it derives methods for drawing inference from observations when these are not exact but subject to variation. As such, the inferences are not exact but probabilistic in nature. Second, it specifies methods for collection of data appropriately so that the assumptions for the application of appropriate statistical methods to them are satisfied. Lastly, techniques for proper interpretation of results are devised.

Experiments and their Designs

An experiment starts with a problem, an answer to which is obtained from interpretation of a set of observations collected suitably. For this purpose a set of experiment units and adequate experimental meterial are required. Equal sized plots of land, a single or a group of plants, etc. are used as experimental units for agricultural experiments. For animal husbandry experiments animals, animal orgeans, etc. form the experimental units. Again, for industrial experiments mechines, ovens and other similar objects form the experimental units.

The problems are usually in the form of comparisons among a set of treatments in respect of some of their effects which are produced when they are applied to the experimental units. For example, in agricultural experiments different variates of a crop, different fertilizer doses, different levels of irrigation, different combinations of level of two or more of the above factors, viz. variety, irrigation, nitrogen fertilizer, date of sowing, etc. may be constitute as the treatments.

Given a set of treatments which can provide information regarding the objective of an experiment, a design for the experiment

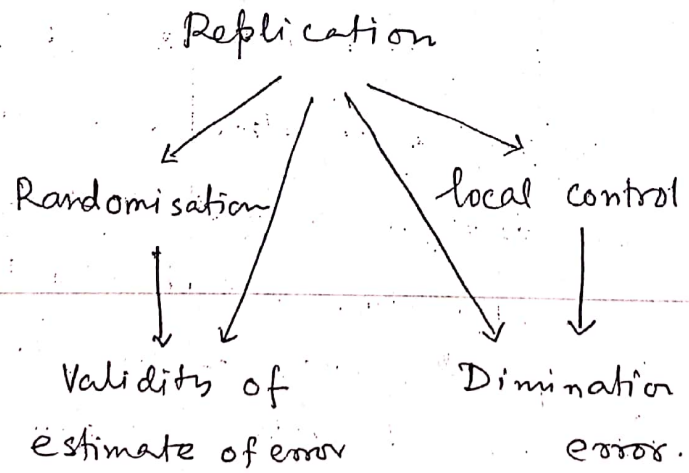
defines the size and number of the experimental units, the manner in which the treatments are allotted to the units and also the appropriate type and grouping of the experimental units. These requirements of a design ensure validity, interpretability and accuracy of the results obtainable from an analysis of the observations. These purposed are served by the priciples of:

(i) randomization, which defines the manner of allocation of the treatments to the experimental units;

(ii) replication, which specifies the number of units to be provided for each of the treatments, and iii) & error control, which increases the precision by choosing appropriate type of experimental units and also their grouping.

Three basic principles of Experimental design

Fisher's diagram



Randomization After the treatments and the experimental units are decided the treatments are allotted to the experimental units at random to avoid any type of personal or subjective bias which may be conscious or unconcious. This ensures validity of the results. It helps to have an objective comparison among the treatments. If also ensures independence of the observations which is necessary for drawing valid inference from the observations by applying appropriate statistical techiques. It is a simple device to achieve the independence of error. Depending on the nature of the experiment and experimental units, there are various experimental designs. Each design has its own way of randomization.

Replications If a treatment is allotted to r exprimental units in an experiment, it is said to be replicated r times. If in a design each of the treatments is replicated r times, the design is said to have r replications. Replication is necessary to increase the accuracy of estimates of the treatment effects. It also provides an estimate of the error variance which is a function of the difference among observations from experimental units under identical treatments.

Replications

Though, the more the number of replications the better it is, so far as precision of estimates in concerned, it cannot be increased indefinitely as it increases cost of experimentation. Moreover, due to limited availability of experimental resources too many replications can not be taken.

The number of replications is, therefore, decided keeping in ~~vein~~ view the permissible expenditure and the required degree of ~~preci~~ precision. Sensitivity of statistical methods for drawing inference also depends on the number of replications in specific experiments.

Experimental Error and Interpretation of Data

After the observations are collected they are statistically analysed so as to obtain relevant information regarding the objective of the experiment.

The objective is usually to make comparisons among the effects of several treatments when the observations are subject to variation. Such comparisons are made by the technique of analysis of variance. Inference is drawn through this technique by comparing two measure of variation, one of which arise due to uncontrolled sources of variation, called the error variation and the other includes variation due to a controlled set of treatments together & with the variance due to all the uncontrolled causes of variation contributing to the error variation.

For example, let there be six plots of land denoted by P_1, P_2, P_3, P_4, P_5 and P_6. The first three plot receives one treatment, say, t_1, and the last three, another treatment, t_2. Suppose, the plots P_1 and P_4 receive one level of irrigation, P_2, P_5 another level and P_3, P_6, a third level. Let y_1, y_2, y_3, y_4, y_5 and y_6 denote the observations on a character recorded from the above six plots in that order. Then the comparison $y_1 - y_2$ which denotes the variation in the observations from the first two plots; does not contain any component of variation due to the treatments as both of them receive the same treatment. But the comparison is not free from the effect of the other controlled factor viz. irrigation as P_1 received on level of irrigation while P_2 received another level. Hence, this comparision by itself does not contribute to error variation. But the comparison $(y_1 - y_2) - (y_4 - y_5)$ is free from the variability caused by both the controlled factors, viz. treatment and irrigation. Hence such comparisons which contain no contribution due to uncontrolled factors like soil fertility and management variation and many others which were not specified in the plots, build up error variance. The actual measure of error variance is a

Determination of number of Replications

Error variance privides a measure of precision of an experiment, the less the error variance the more is the precision. Once a measure of error variance is available for a set of experimental units, the number of replications needed for a desired level of sensitivity can be obtained as below.

Given a set of treatments an experimenter may not be interested to know if two treatments differ in their effects by less than a certain quantity, say, d. In other words, he wants an experiment which should be able to differentiate two treatments when they differ by d or more.

The significance of the difference between two treatments is tested by t-test where

$$t = \frac{\bar{y}_i - \bar{y}_j}{\sqrt{2s^2/r}}$$

Here, $\bar{y}_i$ and $\bar{y}_j$ are the arithmatic means of two treatment effects each based on r replications. s^2 is a measure of the error variation.

Given a difference, d, between two treatment effects such that any difference greater than d should be brought ons as significant by using a design with r replications, the following equation privides a solution of r;

$$t_0 = \frac{|d|}{(\bar{y}_i - \bar{y}_j)/\sqrt{2s^2/r}}$$

where t_0 is the critical value of the t-distribution at the desired level of significance. If σ^2 is known or is based on a very large number of observations, made available from some pilot pre-experiment investigation, then t is taken as the normal variate. If s^2 is estimated with n degrees of freedom then t_0 corresponds to n d.f.

When the number of replications is r or more as obtained above, then all differences greater than d are expected to be brought out as significant by an experiment when it is conducted on a set of experimental units which has variability of the order of s^2.

Another criterion for determining r is to take a number of replications which ensures at atleast 10 d.f. for estimate of error variance in the analysis of variance of the design concerned since the sensitivity of the experiment will be very much lower as the F test which is used to draw inference in such experiments, is very much unstable below 10 d.f.

The above considerations for determining the number of replications hold only for specific designs.

Error control or local control

The considerations in regard to the choice of number of replications ensure reduction of standard error of the estimates of the treatment effects because the standard error of the estimate of a treatment effect is $\sqrt{s^2/r}$. But they cannot reduce the error variance itself, though a large number of replications can ensure a more stable estimate of the error variance. It is, however, possible to devise methods for reducing the error variance. Such measures are called error control or local control. One such measure is to make the experimental units homogeneous. Another method is to form the units into several homogeneous groups, usually called blocks, allowing variation between the group.

Guard Areas

In order to reduce the flow of experimental material from one plot to another, it is customary to leave out strips of land between consecutive plots and also between blocks. These non-experimental areas are known as guard areas.

Block size vs Number of blocks

When the total available experimental area remaining fixed, an increase in the size of plots will decrease the number of blocks or plots. Again if the number of plots increases, the number of guard areas also increases.

Contrasts

The main technique adopted for the analysis and interpretation of the data collected from an experiment is the analysis of variance technique which essentially consists of partitioning the total variance in an experiment into components ascribable to different sources of variation due to the controlled factor and error.

Let $y_1, y_2, \cdots, y_n$ denote n observations or any other quantities. The linear function $C = \sum_{i=1}^{n} l_i y_i$, where l_i's are given number such that $\sum_{i=1}^{n} l_i = 0$ is called a contrast of y_i's.

The restriction $\sum_{i=1}^{n} l_i = 0$, makes C a comparison among the y values.

If $y_1, y_2, \cdots, y_n$ are independent random variables with a common mean μ and variance σ^2, the expected value of the random variable, C is zero and the variance is $\sigma^2 \sum_{i=1}^{n} l_i^2$.

Example $y_1 - y_2 + y_3 - y_4$ is a contrast of y_1, y_2, y_3 and y_4.

$y_1 - 2y + y_3$ is a contrast of y_1, y_2, y_3

Sum of squares (s.s.) of contrasts

The square due to the contrast C is defined as $\frac{C^2}{\sum_{i=1}^{n} l_i^2}$. It is known that this square has a $\sigma^2\chi^2$ chi-square distribution with 1 d.f. when y_i's are normally distributed. Thus, the sum of squares due to two or more contrasts has also a $\sigma^2\chi^2$ distribution if the contrasts are independent.

Multiplication of any contrast by a constant does not materially change the contrast. The square due to a contrast as defined above is not evidently changed by such multiplication.

Orthogonal Contrasts

Two contrasts, $C_1 = \sum_i l_i y_i$ and $C_2 = \sum_i m_i y_i$ are said to be orthogonal if $\sum l_i m_i = 0$

This condition ensures that the covariance C_1 and C_2 is zero when the observations are independent, since

$$\text{Cov}(C_1 C_2) = \text{Cov}\left(\sum l_i y_i, \sum m_i y_i\right)$$

$$= \sum_i l_i m_i \text{Var}(y_i) + \sum_{i \neq j} l_i m_j \text{cov}(y_i, y_j)$$

$$= \sigma^2 \sum_i l_i m_i = 0$$

When there are more than two contrasts, they are said to be mutually orthogonal, if they are pairwise orthogonal.

For example, when there are four observations y_1, y_2, y_3 and y_4 we can write the following three mutually orthogonal contrasts:

i) $y_1 + y_2 - y_3 - y_4$

ii) $y_1 - y_2 - y_3 + y_4$

iii) $y_1 - y_2 + y_3 - y_4$

This means if we use C_1 and C_2 to estimate two different effects, Then the errors in the two estimates will not be related.

The sum of sum of squares due to a set of mutually orthogonal contrasts has a $\sigma^2\chi^2$ distribution with as many degrees of freedom as the number of contrasts in the set.

Maximum number of orthogonal Contrasts

Theorem: Given a set of n values $y_1, y_2, \ldots y_n$, the maximum number of mutually orthogonal contrasts among them is $n-1$.

Proof: Suppose we have the following m mutually orthogonal contrasts:

$$C_j = \sum_{i=1}^{n} l_{ji} y_i \qquad j = 1, 2, \ldots, m$$

Let us now take one more contrast

$C = \sum_{i=1}^{n} l_i y_i$, where l_i's satisfy $\sum l_i = 0$

Now, C can be orthogonal to each of the above m contrasts if the following

simultaneous equations in l_i's have at least one non-zero solution:

$$\sum l_i = 0$$
$$\sum l_{1i} l_i = 0$$
$$\sum l_{2i} l_i = 0$$
$$\vdots$$
$$\sum l_{mi} l_i = 0$$

But this set of homogeneous linear equations in n unknowns can have a non zero solutions, if only if the total number of equations does not exceed $n-1$. Thus m can be at the most $n-2$ and the total number of such contrasts cannot exceed $n-1$. This prove that the maximum number of mutually orthogonal contrasts among n quantites is $n-1$ and that the contrasts can be written in more than one way as there is an infinite number of solutions of the homogeneous equations.

One such $m-1$ set of mutually orthogonal contrasts is:

i) $y_1 - y_2$

ii) $y_1 + y_2 - 2y_3$

iii) $y_1 + y_2 + y_3 - 3y_4$

and so on.

Measures of Variation

The square of a contrast gives a measures of variation due to the contrast. The sum of squares due to the $n-1$ mutually orthogonal constrasts of n observations, give the s.s. due to the n observations. This s.s. divided by the number of independent contrasts on which it is based viz. $(n-1)$ give a measure of variation of the observations and is called mean squares (m.s.). The number of independent contrasts on which an s.s. based is called the degree of freedom (d.f.) of the sum of squares (s.s.).

Certain contrasts represented variation due to uncontrolled causes of variation while certain did not. The S.S. due to contrasts which represent variation due to only uncontrolled causes of variation builds up error variance. Again, the ss. due to some other contrasts which are orthogonal to the above error contrasts and represents comparisons among effects of, say, a set of treatments, builds up the treatment S.S.

For illustration, let us take the ~~case~~ following example. ~~The following table~~ The figures, in the following table, in bracket are the observations on wheat yield collected from an actual experiment ~~card~~ with three irrigation treatments, viz.

I_1 : no irrigation,

I_2 : one irrigation at tillering,

I_3 : two irrigation, one at tillering and one at flowering stage with two varietal treatments t_1 and t_2.

Yield Data from the Irrigation Experiments (lb/acre)

Treatments	Irrigation Levels			Marginal Total
	I_1	I_2	I_3	
t_1	y_{11} (837)	y_{12} (804)	y_{13} (843)	$y_{11}+y_{12}+y_{13}$ (2484)
t_2	y_{21} (914)	y_{22} (758)	y_{23} (849)	$y_{21}+y_{22}+y_{23}$ (2521)
Totals	$y_{11}+y_{21}$ (1751)	$y_{12}+y_{22}$ (1562)	$y_{13}+y_{23}$ (1692)	$\sum y_{ij}$ (5005)

Here y_{ij} $(i=1,2,\ j=1,2,3)$ denotes the observation from the ith treatment and jth level of irrigation.

It will be seen that the following two orthogonal contrasts are free from the effects of the two controlled factors viz. treatments and irrigation:

(i) $(y_{11}-y_{21})-(y_{12}-y_{22})$

ii) $(y_{11}-y_{21})+(y_{12}-y_{22})+2(y_{13}-y_{23})$

Each of these contrasts is actually a contrast of contrasts. In a contrast of two observations in the same column, the effect of irrigation is removed. Again by taking contrast of such contrasts the effects of treatments is eliminated.

Evidently, it is not possible to obtain any more error contrast which is orthogonal to each of above two error contrasts. Hence, the s.s. due to error variation, in short, error s.s. can be obtained by adding the squares of these two contrasts. This s.s. has two d.f.

Again, let us take the contrast $(y_1+y_2+y_3)-(y_4+y_5+y_6)$. It is easily seen that this contrast gives a comparision between the effects of the two treatments. This contrast is not affected by the effects of the irrigation levels as they are evenly distributed in the positive and negative parts of the contrast. There is no other contrast orthogonal to the above which also represents purely treatment comparison. This contrast is also orthogonal to each of the two error contrasts presented earlier. We get the treatment s.s. by obtaining the square of the above contrasts and dividing it by the appropriate factor as indicated earlier. It has 1 d.f. On the hypothesis that there is no variation among the treatment effects, the treatment s.s. is distributed as $\sigma^2\chi^2$ with 1 d.f. Thus on the above hypothesis both the error m.s. and treatment m.s. have the same expected value, σ^2. They are independent as they are obtained from orthogonal contrasts. Their ratio, F, can, therefore, be used to provided a test of the hypothesis of equality of the treatment effects.

To complete the analysis of variance, we have yet to account for two & more orthogonal contrasts each of which is orthogonal to the three contrasts already discussed. We can write these two contrasts as below from the irrigation marginal totals:

(i) $(y_{11} + y_{21}) - (y_{12} + y_{22})$

(ii) $(y_{11} + y_{21}) + (y_{12} + y_{22}) - 2(y_{13} + y_{23})$

They represent comparison of irrigation effects and their S.S. gives a measure of variation due to irrigation effects.

Analysis of Variance Table

Sources of variation	Contrasts	S.S.	d.f.	m.s. $= \frac{S.S.}{d.f}$
Irrigation Components	(i) $(y_{11}+y_{21}) - (y_{12}+y_{22})$ $= I_1 = 189$	$I_1^2/4 = 8930.25$	1	8930.25
	(ii) $(y_{11}+y_{21}) + (y_{12}+y_{22}) - 2(y_{13}+y_{23})$ $= I_2 = -71$	$I_2^2/12 = 420.08$	1	420.08
Irrigation (Total)		$\frac{I_1^2}{4} + \frac{I_2^2}{12} = 9350.33$	2	4675.16
Treatments	(i) $(y_{11}+y_{12}+y_{13}) - (y_{21}+y_{22}+y_{23})$ $= T_1 = -37$	$T_1^2/6 = 228.17$	1	228.17
Error Components	(i) $(y_{11}-y_{21}) - (y_{12}-y_{22})$ $= E_1 = -123$	$E_1^2/4 = 3782.25$	1	3782.25
	(ii) $(y_{11}-y_{21}) + (y_{12}-y_{22})$ $- 2(y_{13}-y_{23}) = E_2 = -19$	$E_2^2/12 = 30.08$	1	30.08
Error		$\frac{E_1^2}{4} + \frac{E_2^2}{12} = 3812.33$	2	1906.17
Total		$\sum_{ij} y_{ij}^2 - \frac{(\sum y_{ij})^2}{6}$	5	

The above table show in a compact from of the details of the analysis of variance obtainable from the contrast approach.

Though the above analysis helps to have a clear understanding of the technique and its rationale, it need not be adopted to analyze larger number of observations, as there are simpler methods for obtaining such sum of squares.

Theorem The sum total of the s.s. due to all the $(n-1)$ mutually orthogonal contrasts among n observations is equal to the s.s. obtained by summing the squares of their deviations from the mean.

Proof Write the contrasts as

$$C_1 = \sum L_{1i} y_i$$

$$C_2 = \sum L_{2i} y_i$$

$$\vdots$$

$$C_{n-1} = \sum L_{(n-1)i} y_i$$

where, $\sum_i L_{mi} = 0$ } $m = 1, 2, \ldots, (n-1)$

and $\sum_i L_{mi}^2 = 1$

$$\sum_i L_{mi} L_{m'i} = 0 \quad , \quad m \neq m' = 1, 2, \ldots, (n-1)$$

s.s. of the contrasts $= C_1^2 + C_2^2 + \ldots + C_{n-1}^2$

Take $C_0 = \dfrac{y_1 + y_2 + \ldots + y_n}{\sqrt{n}}$

The the following matrix is an orthogonal matrix:

$$A = \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ L_{(n-1)1} & L_{(n-1)2} & \cdots & L_{(n-1)n} \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \end{bmatrix}$$

So $A^T A = I$, giving $\sum_m L_{mi}^2 = 1$ and

$$\sum_m L_{mi} L_{mi'} = 0 \qquad (i = 1, 2, \ldots n \text{ and } i' = 1, 2, \ldots n)$$

Omitting the last row of A, we get,

$$\sum_{m=1}^{n-1} L_{mi}^2 = 1 - \frac{1}{n}, \quad \sum_{m=1}^{n-1} L_{mi'} L_{mi} = -\frac{1}{n}$$

In $C_1^2 + C_2^2 + \cdots + C_{n-1}^2$, the coefficient of

$$y_i^2 = \sum_{m=1}^{n-1} L_{mi}^2 = 1 - \frac{1}{n} \text{ and}$$

that of $2 y_i y_{i'} = \sum_{m=1}^{n-1} L_{mi} L_{mi'} = -\frac{1}{n}$

Therefore $C_1^2 + C_2^2 + \cdots + C_{n-1}^2 = \left(1 - \frac{1}{n}\right) \sum y_i^2 - \frac{2}{n} \sum_{i<n} y_i y_{i'}$

$$= \sum y_i^2 - \frac{(\sum y_i)^2}{n}$$

Exercise 1. what comparisions do the following contrasts represent?

(i) $y_1 + y_2 - 2y_3$ (ii) $y_1 + 4y_2 - 2y_3 - 3y_4$

2. Write down a complete set of mutually orthogonal contrasts among $y_1, y_2, \ldots y_n$ using only +1 and −1 as coefficients.

Models and Analysis of Variance

The method of analysis decribed was more a deduction from intutive ~~eg:~~ arguments. It did not clearly specify the nature of the treatments or error effects, that is, if they are fixed or random. A statistical model is actually a linear relation of the effects of the different levels of a number of factors involved in an experiment along with one or more terms representing error effects. The effects of any factor can be either fixed or random. For example, the effects of two well defined levels of irrigation are fixed as each irrigation level can be reasonably taken to have a fixed effects. Again, if the variety of a crop is taken as a factor with a number of varieties of the crop as its levels, then the effects of the varieties will be random if these varieties are selected at random from a large number of varieties. The random effects can again belongs to a finite or an infinite population. The error effects are always random and may belong either to a finite or infinite population.

A model in which each of the factors has fixed effects and only the error effects is random is called a fixed effects ~~and some random~~ model. Models in which some factors have fixed effects and some random effects are called mixed models. Again, models where all the factors have random effects are called random models.

In fixed effect models, the main objectives are to estimate the effects, find a measure of variability among the effects of each of the factors and finally find the variability among the error effects. In random effect models the main emphasis is on estimating the variability among the effects of the different factors. The methodology for obtaining expressions of variability is, however, mostly the same in the different models, though the methods appropriate for their testing are different.

A fixed effect model, for, say, two factors is written as below:

$$y_{ijk} = \mu + a_i + b_j + e_{ijk}$$

where y_{ijk} is an observation coming from a unit defined by the levels, i, j, k of the factors involved, a_i is the effect of the i-th level of one factor, say, A and b_j is the effect of jth level of another factor, say, B and e_{ijk} is an error effect which is assumed by to be normally independently distributed with zero mean and a constant variance σ^2. These assumptions regarding behaviour of e_{ijk} are necessary for drawing inference by adopting known statistical methodology. The methodology that is adopted is the analysis

of variance technique by which inference is drawn by applying F-test. For the F test it is necessary that the observations, that is, the error components should by normally and independently distributed with a common variance. Thus, while collecting observations by adopting various designs to be discussed subsequently it has to be ensured that these assumptions are satisfied by the observations, otherwise no valid inference can be drawn from their analysis.

A further assumption that has been made in the model is that the effects are additive. Though often this assumption is satisfied, it is desirable to get it tested preferably in relatively less known situations. A test due to Tukey is available for this purpose.

Though we have presented above a model involving two factors, there can be other type of models depending on the nature of data, that is, the number of controllable factors involved in the data classification. For example, if the data are from the different levels of a single factor, then we call the data as one-way classified data.

In general, if the data belong to the level combinations of m different factors, we call them m-way classified data.

After a model has been fixed, the general methods of analysis takes the following steps.

Let us the model be doneted by

$$y = f(\mu, a_i, b_j, c_k) + e$$

Where y denotes the observations, e, the error effects and $f(\mu, a_i, b_j, c_k)$ a linear functin of the fixed effects of different factors.

Since $e = y - f(\mu, a_i, b_j, c_k)$

is normally and independently distributed, the estimates of the constant effects μ, a_i, b_j, c_k, can be obtained by the maximum likelihood method of estimate which in the present case gives the same estimates as the Gauss-Markov least square methods of estimation.

That is, let $E = \Sigma e^2 = \Sigma \{ y - f(\mu, a_i, b_j, c_k) \}^2 \text{---} (*)$

where, the summation is over the observations.

Then by solving the normal equations obtained by $\frac{\partial E}{\partial \mu} = 0$, $\frac{\partial E}{\partial a_i} = 0$ $(i = 1, 2, \ldots p$ say$)$

$\frac{\partial E}{\partial b_j} = 0$ $(j = 1, 2, \ldots q$ say$)$, $\frac{\partial E}{\partial c_k} = 0$ $(k = 1, 2, \ldots r$, say$)$

we get as many normal equations as the number of constants, or effects. These normal equations are not independent, so, unique solutions for the effects are not available. Solutions are, however, available only when the sum of the effects of each factor is assumed to be zero.

Once the estimates are available as above, we can get the value of E when the estimates of the effects are substituted in (*). The value of E thus obtained is called the error s.s. It is paid to have degrees of freedom equal to N − k where N is the total number of observations and k is the total number of independent normal equations.

The next problem is to obtain the variability due to the effects of a factor and compare it with the error variability. An hypothesis about the effects of this factor is then made. The hypothesis is usually that the effects of the different levels of the factor are equal, that is, each effect is zero by virtue of the fact the their sum is zero. On this hypothesis we get another model in which the effects of the factor under test is absent. On this model also we can get another error s.s., say, E_1. Actually E_1 contains the variability due to the effects of the factors under test and also the error variability. Thus $E_1 - E$ gives a measure of the variability due to effects of the factor under test and is called the s.s. due to the factor. Its d.f. is one less than the number of levels of the factor. Next the m.s. for each of s.s. obtained by dividing the s.s. by their respective d.f. These two m.s.'s are independent and hence their ratio is tested by the variance ratio test.

Two way Classified data

Let the two factors of classification be denoted by A and B at p and q levels respectively. Let n_{ij} denotes the number of observations in the (i,j)th cell. Denoting the kth observation in the (i,j) th cell by y_{ijk}; $k = 1, 2, \ldots, n_{ij}$; $j = 1, 2, \ldots, q$; $i = 1, 2, \ldots p$.

We make the following substitutions:

(i) $\sum_k y_{ijk} = T_{ij}$, the cell total

(ii) $\sum_j T_{ij} = A_i$, marginal total corresponding to the ith level of A.

(iii) $\sum_i T_{ij} = B_j$, marginal total of B.

(iv) $\sum_i A_i = \sum_j B_j = G$, grand total.

(v) $\sum_j n_{ij} = n_{i\cdot}$, $\sum_i n_{ij} = n_{\cdot j}$ and

$$\sum_i n_{i\cdot} = \sum_j n_{\cdot j} = n$$

Orthogonal data: The concept of orthogonality of data is associated with two or higher way classified data. Let

$$\hat{t}_i = \frac{A_i}{n_{i\cdot}}, \quad (i = 1, 2, \ldots, p)$$

and

$$\hat{b}_j = \frac{B_j}{n_{\cdot j}} \quad (j = 1, 2, \ldots, q)$$

denote the marginal means of the factors A and B respectively in a two way classified table of data. If any contrast $\sum_i l_i \hat{t}_i$ of the marginal means of A is orthogonal to any contrast $\sum_j m_j \hat{b}_j$ of the other marginal means, then the data is are called orthogonal; otherwise, they are non-orthogonal.

Analysis of Non-Orthogonal Two-way data

When the data are non-orthogonal the analysis is no longer simple as solution of normal equations is not always available. We take the following model:

$$y_{ijk} = \mu + a_i + b_j + e_{ijk}$$

where y_{ijk} is the random variable corresponding to the observation Y_{ijk}; μ, a_i, b_j ($i=1,2,\ldots,p$, $j=1,2,\ldots,q$) are fixed effects and ~~because particular~~ e_{ijk} is the error component and arises in experimental data. e_{ijk}'s are normally and independently distributed with zero mean and constant variance, σ^2. We assume that the level of A represent the different columns in the two-way table and those of B, the different rows. The normal equations for obtaining the least-square estimates of the effects are

$$(1) \quad n\hat{\mu} + \sum_i n_{i\cdot}\hat{a}_i + \sum_j n_{\cdot j}\hat{b}_j = G$$

$$(2) \quad n_{i\cdot}\hat{\mu} + n_{i\cdot}\hat{a}_i + \sum_j n_{ij}\hat{b}_j = A_i \quad (i=1,2,\ldots,p)$$

$$(3) \quad n_{ij}\hat{\mu} + n_{j\cdot}\hat{b}_j + \sum_i n_{ij}\hat{a}_i = B_j \quad (j=1,2,\ldots,q)$$

Obtaining $\hat{b}_j$ from (3) and putting it in (3) we get,

$$(4) \quad n_{i\cdot}\hat{\mu} + n_{i\cdot}\hat{a}_i + \sum_j n_{ij}\left(\frac{B_j}{n_{\cdot j}} - \hat{\mu} - \frac{1}{n_{\cdot j}}\sum_{m\neq i} n_{mj}\hat{a}_m\right) = A_i$$

This reduce to

$$\left(A_i - \sum_j \frac{n_{ij}B_j}{n_{\cdot j}}\right) = \hat{a}_i\left(n_{i\cdot} - \sum_j \frac{n_{ij}^2}{n_{\cdot j}}\right) - \sum_{m\neq i}\hat{a}_m\left(\sum_j \frac{n_{ij}n_{mj}}{n_{\cdot j}}\right)$$

---- (5)

Substituting Q_i for $\left(A_j - \sum_j \frac{n_{ij} B_j}{n_{\cdot j}}\right)$

C_{ii} for $\left(n_{i\cdot} - \sum_j \frac{n_{ij}^2}{n_{\cdot j}}\right)$

and C_{im} for $\left(-\sum_j \frac{n_{ij} n_{mj}}{n_{\cdot j}}\right)$

also (note that, $C_{im} = C_{mi}$)

the normal equation (5) are written as

$$C_{ii}\hat{a}_i + \sum_{m \neq i} C_{im}\hat{a}_m = Q_i \quad (i = 1, 2, \cdots, p) \quad \cdots (6)$$

These equations are called reduced normal equations. Q_i is called the adjusted total of the ith level of A. The p equations at (6) are not independent, because when these equations are added both the left and right hand sides vanish. That is $\sum Q_i = 0$ and $\sum_i C_{im} = 0$.

As $C_{im} = C_{mi}$; it also follows that the sum of the coefficients of a_i's in each equation at (6) is zero. Obviously then, if $\hat{a}_i\ (i = 1, 2, \cdots p)$ is a set of solutions of (6), then $\hat{a}_i + \theta\ (i = 1, \cdots p)$ where θ is a constant, is also a set of solution. The equations at (6) have no unique solutions. To get unique solutions we impose the restriction $\sum \hat{a}_i = 0$. This implies that $\hat{a}_i$'s are estimated as deviates from their means. Also the restriction need not be $\sum \hat{a}_i = 0$ always. It can be any linear function of $\hat{a}_i$'s other than their contrasts. Such restrictions change θ only.

When a set of solutions of (6) is obtained, we can get the solutions of b_j's from (3).

The error s.s., $E = \sum_{i,j,k} (y_{ijk} - \bar{\mu} - \bar{a}_i - \bar{b}_j)^2$

$$= \sum_{ijk} y_{ijk}^2 - \bar{\mu} G - \sum_i \bar{a}_i A_i - \sum_j \bar{b}_j B_j$$

$$= \sum_{i,j,k} y_{ijk}^2 - \bar{\mu} G_{\cdot\cdot} - \sum_i \bar{a}_i A_i - \sum_j B_j \left(\frac{B_j}{n_{\cdot j}} - \bar{\mu} - \frac{1}{n_{\cdot j}} \sum_m n_{mj} \bar{a}_m \right)$$

$$= \sum_{i,j,k} y_{ijk}^2 - \sum_j \frac{B_j^2}{n_{\cdot j}} - \sum_i \bar{a}_i Q_i \quad \ldots (7)$$

Instead of eliminating $\hat{b}_j$ to get the reduced normal equations in $\hat{a}_i$'s, we could have eliminated $\hat{a}_i$'s and got the reduced normal equations in $\hat{b}_j$'s and finally the error s.s. as the function of $\hat{b}_j$'s. In this situation

$$E = \sum_{i,j,k} y_{ijk}^2 - \sum_i \frac{A_i^2}{n_{i\cdot}} - \sum_j \bar{b}_j R_j \quad \ldots (8)$$

where R_j is the adjusted total of the jth level of B and is given by

$$R_j = B_j - \sum_i \frac{n_{ij} A_i}{n_{i\cdot}}$$

From (7) and (8) we get,

$$\sum_i \frac{A_i^2}{n_{i\cdot}} - \sum_j \frac{B_j^2}{n_{\cdot j}} = \sum_i \bar{a}_i Q_i - \sum_j \bar{b}_j R_j \quad \ldots (9)$$

Next, for getting the s.s. of A we make the hypothesis $a_1 = a_2 = \cdots = a_k = 0$ and get the reduced model $y_{ijk} = \mu + b_j + \epsilon_{ijk}$.

The error s.s on this model is

$$E_1 = \sum_{ijk} y_{ijk}^2 - \sum_j \frac{B_j^2}{n_{\cdot j}}$$

So, the s.s. of $A = E_1 - E = \sum_i \hat{a}_i Q_i$

It is called the adjusted s.s. of A while $\sum_i \frac{A_i^2}{n_{i\cdot}} - G^2/n$ is called its unadjusted s.s.

Once the adjusted s.s of A is got, the adjusted s.s. due to B can be obtained from (9).

We get the following analysis of variance table.

Sources of variation	d.f	s.s	m.s. = s.s / d.f	Variance Ratio (F)
A (adjusted)	$k-1$	$\sum_i \hat{a}_i Q_i$	$\sum \hat{a}_i Q_i / k-1$	
B (unadjusted)	$r-1$	$\sum_j \frac{B_j^2}{n_{\cdot j}}$	$\sum_j \frac{B_j^2}{n_{\cdot j}} / r-1$	$= F$
Error	$n-k-r+1$	by subtraction		
Total	$n-1$	$\sum_{ijk} y_{ijk}^2$		

When interaction is present, the model is

$$y_{ijk} = \mu + a_i + b_j + h_{ij} + e_{ijk}$$

The normal equations for h_{ij}'s are

$$T_{ij} = n_{ij}\left(\hat{\mu} + \hat{a}_i + \hat{b}_j + \hat{h}_{ij}\right) \qquad \begin{matrix} i = 1, 2, \cdots, p \\ j = 1, 2, \cdots, r \end{matrix}$$

Now, the error s.s. is

$$E_2 = \sum_{ijk} y_{ijk}\left(y_{ijk} - \hat{\mu} - \hat{a}_i - \hat{b}_j - \hat{h}_{ij}\right)$$

$$= \sum_{ijk} y_{ijk}^2 - \sum_{ij} \frac{T_{ij}^2}{n_{ij}}$$

On the hypothesis that h_{ij}'s are zero, we have already got the error s.s, viz.

$$E = \sum_{ijk} y_{ijk}^2 - \sum \frac{B_j^2}{n_j} - \sum_i \hat{a}_i Q_i$$

Hence, the interaction s.s. is

$$E - E_2 = \sum \frac{T_{ij}^2}{n_{ij}} - \sum \frac{B_j^2}{n_{.j}} - \sum_i \hat{a}_i Q_i$$

Its d.f. is $(r-1)(k-1)$, if there is at least one observation in each cell; otherwise, it is reduced by the number of cells having no observation.

Estimate of treatment Contrast and its Variance

Let $\sum_i l_i a_i$ be a treatment contrast. We know that the estimate of a_i is a linear function of Q_i's. Hence $\sum l_i a_i$ is estimated by another linear function of Q_i.

Let $\sum_i l_i \hat{a}_i = \sum_i q_i Q_i$

If $\hat{a}_i$'s are available as linear functions of Q_i's, q_i's can be obtained easily.

Substituting for Q_i in $\sum_i q_i Q_i$ from (6), we get $\sum_i l_i \hat{a}_i = \sum_i q_i (c_{i1}\hat{a}_1 + c_{i2}\hat{a}_2 + \cdots + c_{ik}\hat{a}_k)$

Equating coefficients of $\hat{a}_i$ in this identity we get, $l_i = \sum_m q_m c_{mi} \quad (i = 1, 2, \ldots k)$

These equations are the same as the normal equations at (6) excepting that Q_i's are substituted by l_i's and the unknowns $\hat{a}_i$'s have been written as q_i's.

Hence q_i's can be obtained from the solution of the same normal equations.

Now, $\text{var}(Q_i) = V\left(A_i - \sum_j \frac{n_{ij} B_j}{n_{.j}}\right) = C_{ii}\sigma^2$

$$\text{Cov}(Q_i, Q_m) = \text{Cov}\left\{\left(A_i - \sum_j \frac{n_{ij} B_j}{n_{.j}}\right)\left(A_m - \sum_j \frac{n_{mj} B_j}{n_{.j}}\right)\right\}$$

$$= C_{im}\sigma^2$$

$$\text{So, } \text{var}(\Sigma q_i Q_i) = \sigma^2\left(\Sigma q_i^2 C_{ii} + \sum_{i \neq m} q_i q_m C_{im}\right)$$

$$= \sigma^2\left(\sum_i q_i \sum_m q_m C_{im}\right)$$

$$= \sigma^2 \sum_i q_i l_i$$

In particular, $V(\hat{a}_i - \hat{a}_m) = \sigma^2(q_i - q_m)$

where q_i and q_m are the coefficients of Q_i and Q_m respectively in the expression giving the estimate of $\hat{a}_i - \hat{a}_m$.

We have seen that the S.S. of a contrast is the square of the contrast divided by the coefficient of σ^2 in its variance. Hence, the S.S. of the estimate of $\sum_i l_i a_i$ is

$$\Sigma(q_i Q_i)^2 / \sum_i l_i q_i .$$

Completely Randomized Design (CRD)

Designs are usually characterized by the nature of grouping of experimental units and the procedure of random allocation of treatments to the experimental units. In a completely randomized design the units are taken in a single group. As far as possible the units forming the group should be homogeneous. We shall use the word 'plot' for units.

Let there be k treatments in an experiment. Let the ith treatments be replicated r_i times $(i=1,2,\cdots k)$. The total number of experimental units required for the design is $\sum_{i=1}^{k} r_i$.

Normally, the number of replications for different treatments should be equal as it ensures equal precision of estimates of the treatment effects. The actual number of replications is, however, determined by availability of experimental resources and the requirement of precision and sensitivity of comparisons.

There are several methods for random allocation of treatments to the experimental units. Random number table is used for that.

No local control measure is provided in this design excepting that the error variance can be reduced by choosing a homogeneous set of experimental units.

Analysis

This design provides a one-way classified data according to levels of a single factor, 'treatment'

For its analysis the following model is taken:

$$Y_{ij} = \mu + t_i + e_{ij} \qquad \begin{pmatrix} i = 1,2,\ldots k \\ j = 1,2,\ldots r_i \end{pmatrix}$$

Where Y_{ij} is the random variable corresponding to the observation y_{ij} obtained from the jth replicate of the ith treatment, μ is the general mean, t_i is the fixed effect of the ith treatment and e_{ij} is the error component. The error components are independently and normally distributed with 0 mean and constant variance σ^2.

Let $\sum_j y_{ij} = T_i$ be the observation total of the ith treatment.

$$\text{Treatment sum of squares} = \sum_{i=1}^{k} \frac{T_i^2}{r_i} - \frac{G^2}{R}$$

where $G = \sum_{i=1}^{k} T_i$, $R = \sum_{i=1}^{k} r_i$

$$\text{Total sum of squares} = \sum_{ij} y_{ij}^2 - \frac{G^2}{R}$$

Sources of Variation	D.f.	S.S	m.s = $\frac{S.S}{d.f}$	F
Treatments	$k-1$	$\sum_{i=1}^{k} \frac{T_i^2}{r_i} - \frac{G^2}{R}$	S_t^2	S_t^2/S^2
Error (within treatments)	$R-k$	By subtraction	S^2	
Total	$R-1$	$\sum y_{ij}^2 - \frac{G^2}{R}$		

Randomized Block Designs (RBD)

We have seen that in a CRD no local control measure was adopted excepting that the experimental units should be homogeneous. Usually, when experiments require a large number of experimental unit, CRD cannot ensure precision of the estimates of treatment effects. An improvement of the CRD can be obtained by providing error control measures as described below. This design is called randomized block design. or randomized complete block design..

Let there be k treatments, each of the treatments is replicated the same number of times in this design. Let r denote the number of replications of each treatement. Total number of experimental units is, kr. These units are arranged into r groups, each of size k. The error control measure in this design consists of making the units in each of these group homogeneous. These groups are known as blocks and the experimental units in the blocks are known as plots. In agricultural experiments such groups are formed by neighbouring plots of land of desired size, a set of similar trees, etc. In clinical or similar trials where animals like rats, guinea pigs etc. are the experimental units, animals coming

from the same litter may form such groups. In general, such groups are formed b with units having common characteristics which are known to have influence on the variate under study.

The number of blocks in the design is the same as the number of replications. The k treatments are allotted at random to the k plots in each block. This type of homogeneous grouping of experimental units and the random allocation of the treatments separately in each block are the two main characteristic features of randomized block designs. Actual number of replications in the design is determined by the availability of resources and considerations of cost and precision.

The principle of local control is adopted in this design by first forming homogeneous groups of the units and then allotting at random each treatments once in each group. This results in an increase in precision of estimates of the treatment contrasts, due to the fact that error variance which is a function of comparisions within blocks, is smaller because of homogeneous blocks. This type of allocation makes it possible to elliminate from error variance a portion of variation attributable to block differences.

If, however, variation between the blocks is not significantly large, this type of grouping of the units does not lead to any advantage; rather some degrees of freedom of the error variance is lost without any consequential decrease in the error variance. In such situations CRD is preferable.

When the number of treatments is large, say, greater than 10, RBD are not usually suitable because it is often difficult to get homogeneous groups of units of size larger than 10. If, however, homogeneous groups can be formed with larger number of units in certain situations, RBD can still be adopted with larger number of treatments. Plots of very small area as units in agricultural experiments is an example on this point.

Analysis

The data collected from experiments with RBD form a two-way classification, that is, classified according to the levels of two factors, viz, blocks and treatments. There are kr cells in the two-way table with one observation in each cell. The data are orthogonal and therefore the design is called an orthogonal design.

We take the model

$$Y_{ij} = \mu + t_i + b_j + e_{ij} \qquad \begin{pmatrix} i = 1,2,\ldots,k \\ j = 1,2,\ldots,r \end{pmatrix}$$

where Y_{ij} are the random variable corresponding to the observation y_{ij} from ith treatment in the jth block, μ, t_i, b_j are respectively the general mean, effect of the ith treatment and effect of the jth block. These effects are fixed and e_{ij}'s are i.i.d normal with 0 mean and a constant variance σ^2.

Latin Square Designs (LSD)

RBD is improvement over CRD in some sense that it provide error control measures for elimination of block variation. This principle can be extended further to improve RBDs by eliminating more sources of variation. LSD is one such improved design with provision for the elimination of two sources of variation.

Let there be k treatments each replicated k times so that the total number of experimental units required is k^2. Let P and Q denotes two factors whose variabilities are to be eliminated from the experimental error by having a suitable design. Evidently, both these factors should be related to the variate under study so that their variability may influence the variability of the the variate under study. These two are actually the controlled factors. Each of the factors P and Q is taken at level k. The total number of level combinations of the two factors is k^2. The k^2 experimental units are now so choosen that each unit possesses a different level combination of the two factors.

- The principle of 'local control' was used in the RBD by grouping the units in one way i.e. according to the blocks. In LSD local control is used in two-ways.
- LSD is an incomplete three-way layout, where all the three factors rows, columns and treatments are at the same number of levels (k). For a complete there was layout we need m^3 units, but in the LSD we take only m^2 treatments.

Example

In an animal experiment with the object of comparing effects of four feeds, let young calves be the experimental units with their growth rate during a certain period as the variate under study. Let it be intended to eliminate the variation due to breeds and ages of the calves. So breed and age are the two factors P and Q. The calves are, therefore, to come from four breeds and four age groups. The 16 calves required for the experiment should each belong to a different breed-age combination. It is, therefore, necessary that there should be four calves belonging to each breed and each of these four calves should come from a different age group. In agricultural field plot experiment with k treatments the plots are arranged in form of a $k \times k$ squares so that there are k rows and k columns of the plots. Here rows and columns are the two factors, P and Q. Each plot belongs to different row-column combination.

Originally, latin square designs were defined for eliminating the variation of two factors which are generally called row and column. Though it is not necessary that the two factors should always be called row and column, it has become customary to define latin squares by calling rows and columns.

After the experimental units are obtained as specified above, the treatments are allotted to these units in the following way. The k treatments are allotted to the k^2 units in such a manner that the each treatment occurs only once in each row (each level of the factor P) and once in each column (i.e. each level of the factor Q). This requires that each treatment should be replicated k times.

If a two-way table is formed with the levels of the factors P and Q such that the levels of P denotes the rows and the levels of Q denotes the columns, then the latin square designs requires that the treatments should be so allotted to the k^2 cells of this table that each treatment occurs once in each row and once in each column. Such an arrangement is called a latin square of order k.

For example, suppose there are four treatments denoted by A, B, C and D. Then the following arrangement of treatments in a 4×4 square is a latin square design:

Levels of Q	levels of P			
	P_1	P_2	P_3	P_4
q_1	A	B	C	D
q_2	B	C	D	A
q_3	C	D	A	B
q_4	D	A	B	C

This design is effective if the two factors P and Q can cause variability in the variate under study. If one of the factors does not have substantial influence on the variate under study, elimination of its variance may not reduce the error variance. In such a situation a latin square design is no improvent over the RBD. So, unless it is known that both the factors cause sufficient variation in the variate under study, it is better to adopt a RBD. In agricultural experiment if there is soil fertility in two mutually perpendicular directions, then the adoption of a latin square design with rows and column along the direction of fertility gradients, proves useful.

standard Latin square

An $m \times m$ Latin square with the m letters A, B, C, ... in the natural ordering in the first row and in the first column is called a standard Latin square.

Standard latin square

A	B	C	D
B	C	D	A
C	D	A	B
D	A	B	C

non-standard Latin square

A	B	C	D
D	A	B	C
C	D	A	B
B	C	D	A

Randomization

According to the definition of a LSD, treatments can be allotted to the k^2 units in a number of ways. The purpose of randomization is to select one of these squares at random.

The totality of LSDs obtained from a single LSD by permuting the rows, columns and letter is called a transformation set.

Total number of different LSDs in a transformation set is $n!\,(n-1)!$ times the number of standard LSDs in the set. From a standard $n \times n$ LS, we obtain $n!\,(n-1)!$ different LSDs by permutating all the n columns and the $(n-1)$ rows except the first row.

A B C B C A C A B	A C B B A C C B A	B A C C B A A C B
B C A C A B A B C	C A B A B C B C A	C B A A C B B A C
A B C C A B B C A	A C B C B A B A C	B A C A C B C B A
B C A A B C C A B	C A B B C A A B C	C B A B A C A C B

For the purpose of randomization rows and columns of the square are rearranged at random. Both the rows and columns are, therefore, numbered separately. Keeping the first row as such the remaining rows are rearranged and numbered. If the latin square is of order m, then random numbers less than or equal to $m-1$ are selected by consulting a random number table. If the first number chosen is n_1, then the n_1th row of the initial square is written as the second row. If n_2 is the second number and not equal to n_1, then the n_2th row is written as the third row. This procedure is continued till all the rows of the initial square are placed at random to form another square. After row randomization is over, the columns of the row randomized squares are rearranged by following exactly a similar procedure as for row randomization.

Analysis of Latin square design

In Latin square designs there are three factors. These factors are P, Q and the treatments. The data collected from this design are, therefore, analyed as a threeway classified data. Actually, there should have been k^3 observations as there are three factors each at k levels. But because of the particular allocation of the treatments to the cells, there

is only one observation per cell instead of k in the usual three way classified orthogonal data. As a result we can obtain only the sums of squares due to each of the three factors and error sum of squares. None of the interaction sums of squares of the factors can be obtained. Accordingly we take the model,

$$Y_{ijs} = \mu + r_i + c_j + t_s + e_{ijs}$$

where Y_{ijs} denotes the random variable corresponding to the observation y_{ijs} in the ith row, jth column and under sth treatment, μ, r_i, c_j, t_s $(i, j, s = 1, 2, \ldots k)$ are fixed effects denoting in the order the general mean, the row, the column and the treatment effects. The e_{ijs} is the error component, assumed to be independently and normally distributed with zero mean and a constant variance, σ^2.

The analysis is conducted by following a similar procedure as two-way classified data. The different sums of squares are as below:

Let the data be arranged first in a row × column table such that y_{ij} denotes the observation of (i, j) th cell of the table.

Let $R_i = \sum_j y_{ij}$ = ith row total $(i = 1, 2, \ldots k)$

$C_j = \sum_i y_{ij}$ = jth column total $(j = 1, 2, \ldots k)$

T_s = sum of those observations which come from sth treatment = sth treatment observation total

$G = \sum_i R_i = \sum_j C_j$ = grand total.

Correction factor, C.F. $= \frac{G^2}{k^2}$

Treatment sum of squares $= \sum_s \frac{T_s^2}{k} - C.F.$

Row sum of sum of squares $= \sum_i \frac{R_i^2}{k} - C.F.$

Column sum of squares $= \sum_j \frac{C_j^2}{k} - C.F.$

Analysis of Variance of $k \times k$ LSD

Sources of variation	d.f	S.S.	m.s.	F
Rows	$k-1$	$\sum_i \frac{R_i^2}{k} - C.F.$		
Columns	$k-1$	$\sum_j \frac{C_j^2}{k} - C.F.$		
Treatment	$k-1$	$\sum_s \frac{T_s^2}{k} - C.F.$	S_t^2	S_t^2/s^2
Error	$(k-1)(k-2)$	By Subtraction	s^2	
Total	k^2-1	$\sum_{i,j} y_{ij}^2 - C.F.$		

The hypothesis of equal treatment effects is tested by F-test where F is the ratio of treatment mean squares to error mean squares. If F is not significant, treatment effects do not differ significantly among themselves. If F is significant, further studies to test the significance of any treatment can be made.

Repeated LSD

A latin square design is not usually used when the number of treatments is eight or more, because a design of higher order requires too many replications and it may not also be possible to get the required type of experimental units.

Again, if the number of treatments is very small, then also there are difficulties for adopting a latin square design. When there are two different treatments a latin square design cannot be adopted, because from a 2×2 latin square design error variance is not estimable. For a 3×3 latin square design the d.f. for error sum of squares is 2 and for a 4×4 latin square it is 6. In both these cases the d.f for error sum of squares are too small. To make the design more effective in such cases, latin square design may be repeated, that is instead of taking one latin square, a number of, ~~pa~~ say, p latin squares each of the same order, is taken for the experiment. The treatments are the same in each square but each has a seperate set of units and a separate randomization. The data obtained from such repeated latin squares are known as repeated LSD.

Orthogonal LSs

Two latin squares each of the same order, say, r are said to be orthogonal if when one is superimposed on the ~~others~~ other each symbol of one falls on each symbol of the other once and only once ie., every one of the r^2 pairs of symbols occurs once and only once. For example, the following two latin squares are orthogonal.

Square I

A	B	C
B	C	A
C	A	B

Square II

C	B	A
A	C	B
B	A	C

after superimposing

AC	BB	CA
BA	CC	AB
CB	AA	BC

Another examples are

AA	BB	CC
CB	AC	BA
BC	CA	AB

AB	BA	CC
BC	CB	AA
CA	AC	BB

AA	BB	CC
BC	CA	AB
CB	AC	BA

and their transpose.

Exercise How many 3×3 orthogonal pair of for a particular 3×3 LS.

Congruence

Let, a, b are two integer and n be a positive integer then $a \equiv b \pmod{n}$ if and only if $a - b$ is divisible by n i.e. n is a factor of $a-b$.

Eg. $2 \equiv 10 \pmod{4}$, $3 \equiv 10 \pmod{7}$.

$1 \not\equiv 8 \pmod{5}$, $5 \not\equiv 10 \pmod{4}$.

if $a - b$ is divisible by n, we write $n \mid a-b$.

Euler's Conjecture (1782)

There are no two orthogonal Latin squares of order n, where $n \equiv 2 \pmod{4}$

Theorem (G, Tarry, 1899)

There does not exist two orthogonal LS of order 6.

Theorem (Bose, Shrikhande, Parker, 1959)

For any positive integer $n \geq 6$ there are atleast two orthogonal LS of order n.

Theorem For any positive integer n, there is a LS of order n.

Back circulant Latin square is an example of that. Operation table for $(\mathbb{Z}_n, +)$ gives the LS of order n.

Graeco Latin square designs

This is another name for a pair of orthogonal Latin squares, superimposed one upon another, the treatments being represented by Greek letter in one square and Latin letter in the other.

An GLS is actually an incomplete four way Layout with all the factors at the m levels and observations are take on only m^2 of the possible m^4 treatment combinations.

We have seen that by using LSD treatments effects can be estimated by eliminating two sources of variation. A Graeco Latin square design can eliminate three sources of variation.

An orthogonal design by which three sources of variation can be eliminated by using only k^2 units where k is the number of treatments is called $k \times k$ Graeco Latin square design.

A design is said to be orthogonal when the data obtained from it are orthogonal. When all the levels of a controlled factor occur with each level of any other controlled factor in a design, the data obtained from such a design are always orthogonal.

In order to make the data from a Greeco Latin square orthogonal, we have to take another controlled factor R at K levels in addition to the two controlled factor P and Q introduced for the latin square design, such that each level of R occurs with each level of P and also of Q only once. If treatments in an ordinary latin square design are taken to be levels of the factor R, then such an arrangment is provided by the latin square design. In that case we have to allocate treatments to the units such that each treatment occurs once with each level of the controlled factors P, Q and R in K^2 units

For example, let us replace the four treatments symbols A, B, C, D in the example of the latin square design written systematically by the four Greek letters $\alpha, \beta, \gamma, \delta$ respectively. These letters represent the five levels of the factor R. We then get the following arrangement of four levels of each of three factors in 4^2 units.

	Levels of P			
	α	β	γ	δ
Levels of Q	β	γ	δ	α
	γ	δ	α	β
	δ	α	β	γ

Now four treatments denoted by A, B, C, D are to be allotted to the above 16 units such that each treatment occurs once in each row, and each column and with each of the Greek letters.

Mutually orthogonal Latin Squares (MOLS)

$L_1, L_2, \ldots, L_n$ are called MOLS if for all $i \neq j$, L_i and L_j are orthogonal Latin squares.

Theorem For $n = p$, a prime, there are at least $p-1$ mutually orthogonal Latin squares of order n.

Proof We form

$$L_j = \begin{pmatrix} 0 & 1 & \cdots & p-1 \\ j & 1+j & \cdots & p-1+j \\ 2j & 1+2j & \cdots & p-1+2j \\ \vdots & & & \vdots \\ (p-1)j & 1+(p-1)j & \cdots & p-1+(p-1)j \end{pmatrix}$$

for $j = 1, 2, \ldots, p-1$.

where all elements in L_j are reduced modulo p.

The L_j's are LS because,

$$\alpha, \beta, \gamma \in \{0, 1, \ldots, p-1\}$$

$$\beta + \alpha j \equiv \beta + \gamma j \pmod{p}$$

$$\Rightarrow (\alpha - \gamma) j \equiv 0 \pmod{p}$$

$$\Rightarrow p \mid (\alpha - \gamma) j$$

$$\Rightarrow p \mid \alpha - \gamma \quad \text{as} \quad p \nmid j$$

$$\Rightarrow \alpha - \gamma = 0 \Rightarrow \alpha = \gamma$$

Also, $\beta + \alpha j \equiv \delta + \alpha j \pmod{p}$

$\Rightarrow \beta - \delta \equiv 0 \pmod{p}$

$\Rightarrow p \mid \beta - \delta$

$\Rightarrow \beta - \delta = 0$ as $-(p-1) \leq \beta - \delta \leq p-1$

$\Rightarrow \beta = \delta$.

Hence L_i's are Latin square for all i.

L_i and L_j are orthogonal if $i \neq j$ Since, for suppose, (a, b) be a pair which appeared more than once, say in row α and column β as well as in row γ and column δ.

then $\beta + \alpha j \equiv \delta + \gamma j \equiv a \pmod{p}$

$\beta + \alpha i \equiv \delta + \gamma i \equiv b \pmod{p}$

$\Rightarrow \alpha(i-j) \equiv \gamma(i-j) \pmod{p}$

$\Rightarrow (\alpha - \gamma)(i-j) \equiv 0 \pmod{p}$

Since $i \neq j$ and $i, j \in \{1, 2, \ldots, p-1\}$

$\therefore i - j \not\equiv 0 \pmod{p}$

$\therefore \alpha - \gamma \equiv 0 \mod p$

$\Rightarrow \alpha = \gamma$

$\therefore \beta = \delta$

Hence L_i and L_j are orthogonal for $i \neq j$.

Theorem Let $0, 1, a_2, \ldots, a_{q-1}$ be the elements of $\mathbb{F}_q$ (a finite field of order $q = p^e$, for p prime). Then the Latin squares L_i, $1 \le i \le q-1$, form a set of $q-1$ MOLS where,

$$L_i = \begin{pmatrix} 0 & 1 & a_2 & \cdots & a_{q-1} \\ a_i & a_i+1 & a_i+a_2 & \cdots & a_i+a_{q-1} \\ a_i a_2 & a_i a_2+1 & a_i a_2+a_2 & \cdots & a_i a_2+a_{q-1} \\ \vdots & \vdots & \vdots & & \vdots \\ a_i a_{q-1} & a_i a_{q-1}+1 & \cdots & & a_i a_{q-1}+a_{q-1} \end{pmatrix}$$

Moreover, if ζ is a primitive elements of $\mathbb{F}_q$, then the following $q-1$ LS L_i' with $i = 0, 1, 2, \ldots, q-2$ are also MOLS.

$$L_i' = \begin{pmatrix} 0 & 1 & \cdots & a_{q-1} \\ \zeta^i & 1+\zeta^i & \cdots & a_{q-1}+\zeta^i \\ \zeta^{1+i} & 1+\zeta^{1+i} & \cdots & a_{q-1}+\zeta^{1+i} \\ \vdots & \vdots & & \vdots \\ \zeta^{q-2+i} & 1+\zeta^{q-2+i} & \cdots & a_{q-1}+\zeta^{q-2+i} \end{pmatrix}$$

L_{i+1}' can be obtained from L_i' by a cyclic exchange of the last $q-1$ rows.

Proof Exercise. (similar to above theorem)

Result Let $N(n)$ denote the maximum number of MOLS of order n, with the convention that $N(1) = \infty$.

Since a Latin square exists for every order n, $N(n) \geq 1$.

Also $N(2) = 1$, $N(3) = 2$, $N(6) = 1$., $N(10) \geq 2$

$N(n) \geq 3$ $\quad n \notin \{2, 3, 6, 10\}$.

Number of standard LS for $m = 3$ is 1

" " " " for $m = 4$ is 4.

" " " " for $m = 5$ is 56.

Theorem $N(nm) \geq \min\left(N(m), N(n)\right)$ for $m, n > 1$.

Proof We will first show if there is a pair of MOLS of order n and a pair of MOLS of order m, then there is a pair of MOLS of order mn.

Let A_1 and A_2 be MOLS of order n and B_1 and B_2 MOLS of order n, then $A_1 \otimes B_1$ and $A_2 \otimes B_2$ are MOLS of order mn.

Where $A_i \otimes B_i$ are kronecker product of A_i and B_i, for $i = 1, 2$.

If $A_1 = (a_{ij}^{(1)})$ then

$$A_1 \otimes B_1 = \begin{pmatrix} a_{11}^{(1)} B_1 & \cdots & a_{1m}^{(1)} B_1 \\ \vdots & & \\ a_{m1}^{(1)} B_1 & \cdots & a_{mm}^{(1)} B_1 \end{pmatrix}$$

Now, $a_{ij}^{(1)} b_{jk}^{(1)} = a_{\alpha\beta}^{(1)} b_{\gamma\delta}^{(1)}$

and $a_{ij}^{(2)} b_{jk}^{(2)} = a_{\alpha\beta}^{(2)} b_{\gamma\delta}^{(2)}$

$\Rightarrow a_{ij}^{(1)} = a_{\alpha\beta}^{(1)}$, $b_{jk}^{(1)} = b_{\gamma\delta}^{(1)}$, $a_{ij}^{(2)} = a_{\alpha\beta}^{(2)}$, $b_{jk}^{(2)} = b_{\gamma\delta}^{(2)}$

$\Rightarrow i = \alpha$, $j = \beta$, $l = \gamma$, $k = \delta$

Exercise

1. Obtain the layouts of a CRD with three treatments, A, B and C, the replication numbers being 6, 5 and 10 respectively.

2. Obtain the layouts of an RBD with five treatments in four blocks.

3. An $r \times n$ latin rectangle is an $r \times n$ array made out of the integers $\{1, 2, \ldots, n\}$ such that no integer is repeated in any row or in any column. Given an example of a 3×5 Latin rectangle with $r \leq n$ and extend it to a Latin square of order 5. Can this Latin rectangle be extended to a Latin square?

4. Let $L^{(k)} = (a_{ij}^{(k)})$ where $a_{ij}^{(k)} = i + jk \pmod 9$. Which of $L^{(k)}$, $1 \leq k \leq 8$, are Latin squares? Are $L^{(2)}$ and $L^{(5)}$ orthogonal?

5. Show that for $n \geq 2$ there can be at most $n-1$ mutually orthogonal Latin squares of order n.

6. Let $A = (a_{ij})$ and $B = (b_{ij})$ be two orthogonal Latin squares of order n with entries in $\{0, 1, \ldots, n-1\}$ such that the sum of entries in each of diagonals of A and B is $n(n-1)/2$. Show that $M = (n a_{ij} + b_{ij} + 1)$ is a magic square of order n. Construct a magic square of order 4 from two orthogonal LS.

7. Take two pair of MOLS of order 3 and use the Kronecker product to construct a pair of MOLS of order 12.

Factorial Experiment

There are several types of experiments which require statistical investigation. These are characterized by the nature of treatments under investigation and also the nature of comparison required among them so as to meet the objectives of the experiment. There are three main types of experiments: (i) varietal trials (ii) factorial experiments and iii) bio-assays.

In factorial experiments combinations of two or more levels of more than one factor are the treatments. If there are p different levels of one factor and q different levels of another factor viz. p different varieties or levels of the factor fertilizer 'Potash' and q different levels of the factor fertilizer 'Nitrogen' and if if experiment conducting such a way that pq different treatment combinations are used to investigate the variation in two factors simultaneously then that experiment is known as $p \times q$ factorial experiment.

If there are n factors, each at two levels then the experiment is known as 2^n factorial experiment.

If there are n factors, each at three different levels then the experiment is known as 3^n factorial ~~desi~~ experiment. More generally if there are n factors, each at s - different levels then the experiment is known as s^n factorial experiment.

If the number of levels of each factor in an experiment is the same, the experiment is called symmetrical factorial; Otherwise, it is called asymmetrical factorial or sometimes called mixed factorial.

For example, with two factors (i) nitrogen fertilizer at three levels, denoted by n_0, n_1 and n_2 and (ii) irrigation at two levels, I_0 and I_1 in an agricultural experiment we can form the following six combinations taking one level from each factor $I_0n_0, I_0n_1, I_0n_2, I_1n_0, I_1n_1$ and I_1n_2. Such combinations form treatments in factorial experiments. This is an example of 3×2 asymmetrical factorial experiment.

The comparisons required in this type of experiments are not the pair comparisons as in varietal trials but comparison of main effects and interactions. The contrast obtained from the difference of the totals of the first three and the last three of the above six combinations gives the main effects contrast for irrigation. This is so, because the total of the first three treatment combinations represents the effect of irrigation at its I_0 levels while the sum of the last three represents its effect at I_1 level, of course, in each case in presence of same levels of nitrogen. Hence, there difference gives a

mparison between the responses from the two levels of irrigation. This comparison, therefore, represents the main effect of irrigation.

Similarly, two independent, comparisons among the totals, $I_0n_0 + I_1n_0$, $I_0n_1 + I_1n_1$ and $I_0n_2 + I_1n_2$ represent the main effect of nitrogen.

Again, we can get an effect of irrigation in presence of, say, n_0 from $I_0n_0 - I_1n_0$. Two more such effects can be obtained from the other two levels of nitrogen, viz. $I_0n_1 - I_1n_1$ and $I_0n_2 - I_1n_2$. Comparison among these three contrasts indicates the equality or otherwise of the effects of irrigation at the different different levels of nitrogen. This sort of comparison, therefore, indicates if the factors act independently or they interact to influence the yield of the experimental unit on which they appear simultaneously. Contrasts representing such comparisons, that is, $(I_0n_0 - I_1n_0) - (I_0n_1 - I_1n_1)$ are called interaction effects. In this example there are in all two interaction contrasts.

To conduct an experiment by adopting a RBD so as to have at least 10 d.f. for error variance, we require six 6 replications for the experiment on nitrogen alone and 11 replication for experiment on irrigation. Therefore, a total 40 experimental units are required. But for conducting a factorial experiment with six combinations of two factors as treatments, we require 18 units only.

2^2 - factorial experiment

Let A_1, A_2 are two levels of a factor A and B_1, B_2 are two levels of another factor B. The four treatment combinations A_1B_1, A_1B_2, A_2B_1, and A_2B_2 can be compared by lying out the experiment in CRD, RBD or LSD.

RBD of four treatments may be as follows.

A_1B_1	A_1B_2	A_2B_1	A_2B_2

block 1 (replication-1)

A_2B_1	A_1B_1	A_2B_2	A_1B_2

block 3 (replication-3)

A_1B_1	A_2B_1	A_1B_2	A_2B_2

block 2 (replication-2)

A_1B_1	A_2B_2	A_1B_2	A_2B_1

block 4 (replication-4)

CRD of four treatments as follows:

A_1B_1	A_1B_2	A_1B_1	A_2B_1
A_2B_1	A_2B_2	A_1B_2	A_2B_1
A_2B_2	A_1B_1	A_2B_2	A_1B_2
A_2B_1	A_2B_2	A_1B_1	A_1B_2

LSD of four treatment may as follows

A_1B_1	A_1B_2	A_2B_1	A_2B_2
A_1B_2	A_2B_1	A_2B_2	A_1B_1
A_2B_1	A_2B_2	A_1B_1	A_1B_2
A_2B_2	A_1B_1	A_1B_2	A_2B_1

In factorial experiment our main objective is to carry out seperate tests for the main effects A, B and the interaction AB.

The treatment combinations A_1B_1, A_1B_2, A_2B_1 and A_2B_2 sometimes written like either I, a, b, ab or 00, 01, 10, 11. The symbol I i.e. 00 denotes that both the factors are at the lower levels in this combination. It is also called the control treatment.

Let [1], [a], [b], [ab] denote the total yields of the r-units (plot) receiving the treatments I, a, b and ab respectively. Let the corresponding average obtained on dividing these totals by r be denoted by (1), (a), (b) and (ab) respectively. The letters M, A, B and AB are the mean yield, effect due to a, effect due to b and their interaction respectively.

$$A = \frac{1}{2}\left[\{(ab)-(b)\}+\{(a)-(1)\}\right]$$

$$= \frac{1}{2}\left[(ab)-(b)+(a)-(1)\right]$$

$$= \frac{1}{2}(a-1)(b+1)$$

$$B = \frac{1}{2}\left[\{(ab)-(a)\}+\{(b)-(1)\}\right]$$

$$= \frac{1}{2}\left[(ab)-(a)+(b)-(1)\right]$$

$$= \frac{1}{2}(a+1)(b-1)$$

$$M = \frac{1}{4}\left[(ab)+(ab)+(b)+(1)\right] = \frac{1}{4}(a+1)(b+1)$$

$$AB = \frac{1}{2}\left[\{(ab)-(b)\}-\{(a)-(1)\}\right]$$

$$= \frac{1}{2}\left[(ab)-(b)-(a)+(1)\right]$$

$$= \frac{1}{2}(a-1)(b-1)$$

Note that, $BA = \frac{1}{2}\left[\{(ab)-(a)\}-\{(b)-(1)\}\right]$

$$= AB.$$

Table of signs and divisor giving M, A, B, AB

Factorial effects	Treatment mean (1)	(a)	(b)	(ab)	Divisors
M	+	+	+	+	4
A	−	+	−	+	2
B	−	−	+	+	2
AB	+	−	−	+	2

Note that, A, B and AB are mutually orthogonal contrast.

Also note that, if we laying out CRD, RBD or LSD just using different levels of two factors A_1, A_2, B_1, B_2 the we cannot estimate main effect and interaction.

Yate's method for finding effects

Treatement combination	Total		
(1)	[1]	[1] + [a]	[1] + [a] + [b] + [ab]
a	[a]	[b] + [ab]	[a] − [1] + [ab] − [b]
b	[b]	[a] − [1]	[b] + [ab] − [1] − [a]
ab	[ab]	[ab] − [b]	[ab] − [b] − [a] + [1]

Analysis of variance for 2^2 factorial

There are 3 d.f. for treatment variation have bee divicled into three components A, B and AB each having 1 d.f. The first two components are the main effects of the factors A and B respectively and the third component is their interaction ~~are is~~

Sources	d.f	S.S
Replications	$r-1$	
Treatments	3	
A	1	$(-[1] - [b] + [a] + [ab])^2/4r$
B	1	$(-[1] + [b] - [a] + [ab])^2/4r$
AB	1	$(.[1] - [b] - [a] + [ab])^2/4r$
Error	$3(r-1)$	
Total	$4r-1$	

As the three contrasts are mutually orthogonal, we can get the treatment S.S from the total of their sum of squares

2^3 - Factorial experiment

When there are three factors each at two levels, the factorial is denoted by 2^3 as there are eight treatment combination. These are written as

$a_0b_0c_0$, $a_0b_0c_1$, $a_0b_1c_0$, $a_0b_1c_1$, $a_1b_0c_0$, $a_1b_0c_1$, $a_1b_1c_0$, $a_1b_1c_1$

or 000, 001, 010, 011, 100, 101, 110, 111

or 1, c, b, bc, a, ac, ab, abc.

Main effect of A

Level of B	Level of C	Simple effect of A
b_0	c_0	(a) − (1)
b_1	c_0	(ab) − (b)
b_0	c_1	(ac) − (c)
b_1	c_1	(abc) − (bc)

The main effect of A is defined as the average of thes four simple effects

Thus $$A = \frac{1}{4}[(a) - (1) + (ab) - (b) + (ac) - (c) + (abc) - (bc)]$$

$$= \frac{1}{4}(a-1)(b+1)(c+1)$$

Similary the main effects of B is

$$B = \frac{1}{4}(a+1)(b-1)(c+1)$$

and the main effect of C is

$$C = \frac{1}{4}(a+1)(b+1)(c-1)$$

ne average effect of A at the level b_0 of B is

$$\frac{1}{2}[(ac)-(c)+(a)-(1)] = \frac{1}{2}(a-1)(c+1)$$

and the average effect of A at the level b_1 of B is $\frac{1}{2}[(abc)-(bc)+(ab)-(b)] = \frac{1}{2}(a-1)(c+1)(b)$

If the factors A and B are independent then we would expect that the average effect of A will remain the same at either level of B. But if the factors A and B are not independent then measure of interaction AB is given by

$$AB = \frac{1}{2}\left[\frac{1}{2}\{(abc)-(bc)+(ab)-(b)\} - \frac{1}{2}\{(ac)-(c)+(a)-(1)\}\right]$$

$$= \frac{1}{4}[(abc)-(bc)+(ab)-(b)-(ac)+(c)-(a)+(1)]$$

$$= \frac{1}{4}(a-1)(b-1)(c+1)$$

Similarly, $AC = \frac{1}{4}(a-1)(b+1)(c-1)$

and $BC = \frac{1}{4}(a+1)(b-1)(c-1)$

Interaction of AB at level c_0 of C is

$$\frac{1}{2}[(ab)-(b)-(a)+(1)]$$

and interaction of AB at level c_1 of C is

$$\frac{1}{2}[(abc)-(bc)-(ac)+(c)]$$

Thus interaction effect of AB with C, ie. the interaction ABC is given by

$$\frac{1}{2}\left[\frac{1}{2}\{(abc)-(bc)-(ac)+(c)\} - \frac{1}{2}\{(ab)-(b)-(a)+(1)\}\right]$$

$$= \frac{1}{4}[(abc)-(bc)-(ac)+(c)-(ab)+(b)+(a)-(1)]$$

$$= \frac{1}{4}(a-1)(b-1)(c-1)$$

Main effect and Interaction Contrasts in 2^3 Factorials

Contributions	A	B	C	AB	AC	BC	ABC
(1)	−	−	−	+	+	+	−
(c)	−	−	+	+	−	−	+
(b)	−	+	−	−	+	−	+
(bc)	−	+	+	−	−	+	−
(a)	+	−	−	+	−	+	+
(ac)	+	−	+	−	+	−	−
(ab)	+	+	−	+	−	−	−
(abc)	+	+	+	+	+	+	+

Remarks

1. The seven effects are mutually orthogonal contrasts of the treatment means.
2. The s.s. due to any effects is obtained by first getting its contrast value from the treatment totals and then dividing its square by the total number of observations.
3. As the contrasts are mutually orthogonal the sum total of their s.s. gives the treatment s.s with d.f. 7.
4. It can be shown that,

$$AB = BA \;,\; AC = CA \;,\; BC = CB$$

and $ABC = ACB = BCA = BAC = CAB = CBA$ etc.

The set of letters $\{1, A, B, C, AB, AC, BC, ABC\}$ form an abelian group under the composition 'o' where 'o' is defined as follows.

$1 \circ 1 = 1$, $1 \circ A = A \circ 1 = A$, $A \circ A = A^2 = 1$, $A \circ B = B \circ A = AB$

$A \circ AB = A^2 \circ B = B$, $AB \circ AB = 1$, $AB \circ ABC = C$,

$ABC \circ ABC = 1$ etc.

1 is the identity element and each elements has its own inverse.

Yate's method for computing the effects

Treatment Combinations	Total			Effects
(1)	[1]	[1] + [a]	[1] + [a] + [b] + [ab]	(i) + (ii)
a	[a]	[b] + [ab]	[c] + [ac] + [bc] + [abc]	(iii) + (iv)
b	[b]	[c] + [ac]	[a] − [1] + [ab] − [b]	(v) + (vi)
ab	[ab]	[bc] + [abc]	[ac] − [c] + [abc] − [bc]	(vii) + (viii)
c	[c]	[a] − [1]	[b] + [ab] − [1] − [a]	(ii) − (i)
ac	[ac]	[ab] − [b]	[bc] + [abc] − [c] − [ac]	(iv) − (iii)
bc	[bc]	[ac] − [c]	[ab] − [b] − [a] + [1]	(vi) − (v)
abc	[abc]	[abc] − [bc]	[abc] − [bc] − [ac] + [c]	(viii) − (vii)

Remark It has been seen that with the help of each contrast of a main effect or interaction the treatment combinations can be divided into two groups One group is formed of those which occur with negative sign in the contrast and the other, of what occur with positive sign. The interaction or main effect is the difference between these two group means

Model of 2^3 design

$$y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \rho_l + \epsilon_{ijkl}$$

where $i, j, k = 0, 1$, $l = 1, 2, \ldots r$.

μ = general mean, α_i, β_j and γ_k are effects of the ith level of A, jth level of B and kth level of C respectively. $(\alpha\beta)_{ij}$ is the interaction effect of ith level of A with jth level of B, $(\alpha\gamma)_{ik}$ and $(\beta\gamma)_{jk}$ similarly defined. $(\alpha\beta\gamma)_{ijk}$ is the interaction effect of ith level of A with jth level of B and kth level of C, ρ_l is the effect due to the l-th replicate. ϵ_{ijkl} represent the error effect which are i.i.d $N(0, \sigma_e^2)$.

The above parameters are subject to the following conditions. $\sum \alpha_i = \sum \beta_j = \sum \gamma_k = \sum \rho_l = 0$

and $\sum (\alpha\beta)_{ij} = \sum (\beta\gamma)_{jk} = \sum (\alpha\gamma)_{ik} = \sum (\alpha\beta\gamma)_{ijk} = 0$

2^n - Factorial Experiment

Suppose A, B, C, --- K are the factors each at two levels 0 and 1. Corresponding small letters a, b, c, - -, k denote the corresponding factors at the second level; the first level of any factor being signified by the absence of the corresponding small letters. Absence of all letter will be denoted by 1. or (1).

also for n factors $A, B, C, \cdots K$ the main effects and interactions are given by the expression :

$$\frac{1}{2^{n-1}} \left[(a \pm 1)(b \pm 1) \cdots (k \pm 1)\right].$$

the corresponding sign in each factor being taken as negative if the corresponding factor is contained in the factorial effect. As usual, the expression is to be expanded algebraically and then the treatment combinations are to be replaced by the corresponding treatment means.

Remarks 1. Yate's method can be apply similarly as 2^2 and 2^3.

2- Group structure can be construct as in 2^2 and 2^3.

Experiments with factors at three levels each (3^2- experiment)

When factors are taken at three levels instead of two, the scope of the experiment increases. It becomes more informative. Further, when the levels of a factor are quantitative, the pattern of change of response with the increase of the level values of the factor can be studied in a better way. A study to investigate if the changes is linear or quadratic is possible when the factors are at three levels, through from this point of view the more the numbers of levels the better. However, the number of levels of the factors cannot be increased too much as the size of the experiment increases too rapidly.

Let us begin with two factors A and B each at three levels. We shall use 0, 1 and 2 as the level codes and treat them as the three elements of mod 3. In future, all mathematical treatments on these elements will be in the finite field.

The 9 treatments are

00, 01, 02, 10, 11, 12, 20, 21, 22

Among these 9 treatments three will be 8 comparisons (2 main effects of A, 2 main effect of B, 4 interaction effect of AB)

The s.s due to A is obtained by first grouping the 9 treatments into three groups. The first group contains all those treaments in which A occurs at 0 level. The second and third groups contain treatments having the level of A as 1 and 2 respectively. A comparison among these three group totals, give comparison denoting the main effect of A. As there are two independent contrasts among the three group totals, the d.f of the main effect of A is 2. The s.s. can be obtained either from the total of the s.s. due to the two contrasts or from the three group totals by the usual method of deviation squares.

Component denoted by	Defining equations
$[A]_i$	$x_1 = i$
$[B]_i$	$x_2 = i$
$[AB]_i$	$x_1 + x_2 = i$
$[AB^2]_i$	$x_1 + 2x_2 = i$
$[A^2B]_i$	$2x_1 + x_2 = i$
~~$[A^2B^2]_i$~~	
$[A^2B^2]_i$	$2x_1 + 2x_2 = i$

for $i = 0, 1, 2$ all the numbers being reduced modulo 3. For example $[AB]_0 = [00] + [12] + [21]$.

Analysis of variance

Sources	d.f.
Replications	$r-1$
Treatments	8
A	2
B	2
AB	2
AB^2	2
Error	$8(r-1)$
Total	$9r-1$

Table of sign and divisors for calculating linear and quadratic components

Treatment Effects	Treatment Total [00]	[01]	[02]	[10]	[11]	[12]	[20]	[21]	[22]	Division by
A_L	−	−	−	0	0	0	+	+	+	6r
A_Q	+	+	+	−2	−2	−2	+	+	+	18r
B_L	−	0	+	−	0	+	−	0	+	6r
B_Q	+	−2	+	+	−2	+	+	−2	+	18r
$A_L \times B_L$	+	0	−	0	0	0	−	0	+	6r
$A_L \times B_Q$	−	2	−	0	0	0	+	−2	+	12r
$A_Q \times B_L$	−	0	+	2	0	−2	−	0	+	12r
$A_Q \times B_Q$	+	−2	+	−2	4	−2	+	−2	+	36r

A 3^3- experiment in blocks of 9 plot each

In this experiment all possible combination will give rise to 27 treatments of the form (x_1, x_2, x_3) where x_1, x_2, and x_3 are the level of A, B and C respectively.

Defining equation for component would be same as in 3^2 design. for example to obtain the total of $[AB^2C]_1$ the defining equation is

$$x_1 + 2x_2 + x_3 = 1 \pmod 3$$

i.e. $[AB^2C]_1 = [001] + [100] + [012] + [111] + [202] + [210] + [020] + [122] + [221]$

Source	d.f
Replicates	$r-1$
Treatments	
Main effects	
A	2
B	2
C	2
2-factor interaction	
A × B	4
A × C	4
B × C	4
3-factr interaction	
A × B × C	8
Error	$26(r-1)$
Total	$27r-1$

Confounding in factorial Designs:

Confounding in experimental designs is an arrangement of the treatment combinations in the blocks in which less important treatment effects are purposively mixed up or entangled with the blocks effects.

It is the technique of reducing the size of replication over a number of blocks at the cost of losing some information on some effect which is less important or not of much practical importance.

In factorial experiments as the number of factors and/or that of levels of the factors increases and hence the block size has to be enlarged. Therefore heterogeneity introduced as

a consequence of the size of the experiment for this reason principle of local control should be apply. Confounding is one such technique.

Orthogonality and Confounding

Let X_i, $i = 1, 2, \ldots n$ be i.i.d r.v's distributed as $N(0, \sigma^2)$. Let us consider their two orthogonal contrast U and V defined as follows.

$$U = \sum_{i=1}^{n} \lambda_i X_i \quad , \quad V = \sum_{i=1}^{n} \mu_i X_i$$

$\sum \lambda_i = \sum \mu_i = \sum \lambda_i \mu_i = 0$.

Then U and V are independently and normally distributed.

According to F. Yates, "Orthogonality of a design is the property which assures that different effects will be capable of separate estimation and testing without any entanglement."

In an orthogonal design there are no problems of independent estimation of various effects and their test of significance.

For example CRD, RBD and LSD are orthogonal designs. The deliberate introduction of non-orthogonality in a design, in order to get better estimates and test on important comparisons is called confounding.

Confounding in a 2^3-experiment

Replicate

Block 1	Block 2
a	ab
b	bc
c	ca
abc	(1)

ABC Confounded.

Block 1 contain all + sign and Block 2 contain all − sign of the contrast due to effect of ABC. If each block contain same number of + and − sign of the contrast due to any effect, then that effect do not confounded with the block effects. Usually we confounde higher order interaction as they are sometimes zero or very small.

Complete are partial Confounding

In complete confounding, we confound the same interaction in all the replications. When an interaction is confounded in one replicate and not in another the experiment is said to be parially confounded.

For example:

Block 1	Block 2
a	ab
b	bc
c	ca
abc	(1)

Replicate (1)

Block 3	Block 4
a	ab
b	bc
c	ca
abc	(1)

replicate 2

Block 5	Block 6
a	ab
b	bc
c	ca
abc	(1)

replicate 3

Total confounding of ABC in 6 block with 3 replicates.

Partial confounding of AB, BC and CA in 6 blocks with 3 replicates.

Replicate 1		Replicate 2		Replicate 3	
Block 1	Block 2	Block 3	Block 4	Block 5	Block 6
ab	a	bc	b	ca	c
abc	b	abc	c	abc	a
c	ac	a	ab	b	ab
(1)	bc	(1)	ac	(1)	bc
AB confounded		BC confounded		CA confounded	

In total confounding of ABC information about ABC losses 100% in but in partial confounding information losses of AB, BC and CA is $\frac{100}{3}$% each.

A 2^n-experiment in 2^k blocks per replicate

Here each block will receive 2^{n-k} treatment combinations. In this experiment in a replicate confounds $2^k - 1$ factorial effects with blocks.

The particular set of $2^k - 1$ effects that is confounded depends on the layout of that replicate. Depending on whether we have the same layout in each replicate or different layouts for different replicates, we have complete or partial confounding, respectively.

Since there 2^k blocks, number of plot needed in each block is $2^n / 2^k = 2^{n-k}$.

Of the $2^k - 1$ factorial effects, we are confounded in a replicate, we may select k factorial effects as we please, subject to the restriction that none of these should be a generalised interaction of the others included in this set of k effects.

The generalised interaction of two or more effects is the effect that is obtained by combining the letters of the effects and neglecting a letter if it occurs twice, we may commute the letters.

E.g. The generalised interaction of ABC, BCDF and AEF is $ABC\ BCDF\ AEF = A^2B^2C^2DEF^2 = DE$

It can be shown that if in a replicate two interactions are confounded then their generalised interactions is also confounded.

So from k effects we can find remaining $2^k - 1 - k$ effects by generalised interaction.

Hence, all the letters combination which confounded and (1) forms a subgroup of order 2^k of the abelian group of order 2^n with the generating set of k effect which we select first.

Group of treatment letters

Consider the set of all treatments with (1). Define a binary operation '*' on this set as follows if T_1 and T_2 are two treatment combinations, then $T_1 * T_2$ is the generalised interaction of T_1 and T_2.

Under this operation the set of all treatments & gen combinations forms an abelian group or commutative group with identity element (1) and the inverse of an element is itself. So it is a abelian group of order 2^n and it is a vector space over field Z_2.

$Z_2 = \{0, 1\}$ with $0+0=0$, $0+1=1+0=1$

$1+1=0$, $0.0 = 0.1 = 1.0 = 0$, $1.1 = 1$.

Independent Treatments

We say $T_1, T_2, \ldots T_k$ are independent treatments if $T_1^{c_1} T_2^{c_2} \ldots T_k^{c_k} = (1)$, where $c_1, c_2, \ldots c_k$ are either 0 or 1, has unique solution

$c_1 = c_2 = \ldots = c_k = 0$.

Eg. ABC, BCDF and AEF are independent because if $c_1, c_2, c_3 \in \{0, 1\}$ and

$(ABC)^{c_1} (BCDF)^{c_2} (AEF)^{c_3} = (1)$

then $A^{c_1+c_3} B^{c_1+c_2} C^{c_1+c_2} D^{c_2} E^{c_3} F^{c_2+c_3} = (1)$

$\Rightarrow c_1 + c_3 = c_1 + c_2 = c_2 = c_3 = c_2 + c_3 = 0$

$\Rightarrow c_1 = c_2 = c_3 = 0$

To construct generalised interaction from these three independent treatments we use the group operation.

The generalised interactions are

$(ABC)(BCDF) = ADF$

$(ABC)(AEF) = BCEF$

$(BCDF)(AEF) = ABCDE$

$(ABC)(BCDF)(AEF) = DE$

Note that, ABC, AEF and BCEF are not independent as $(ABC)^1 (AEF)^1 (BCEF)^1 = (1)$.

Note that, the treatment combinations ABC, BCDF, AEF, ADF, BCEF, ABCDF and DE along with the treatment combination (1) forms a vector space of dimension 3 over the field Z_2. Here the set of three independent treatment combinations or vectors forms a basis which is also known as generating set.

Layout of 2^n-experiment in 2^k blocks

To get the layout of a 2^n- experiment in 2^k blocks in a replicate, we first decide which treatment combinations, we want to confound in this replicate. Then we form the intrablock subgroup or principal block of the replicate.

It is that block which contains the treatments (1) and other $2^{n-k} - 1$ treatment combinations each having an even number of letters (including no letters) in common with each of factorial effects confounded in that replicate.

For example, Consider the 2^4 factorial design in 2^2 blocks, Hence there are 16 treatment combination $A_0^{C_1} B^{C_2} C^{C_3} D^{C_4}$ where $C_1, C_2, C_3, C_4 \in \{0, 1\}$ and $A^0 B^0 C^0 D^0$ means the treatment combination (1).

Now there are $2^2 - 1 = 3$ treatment combinations which confound with blocks among these three treatment combinations we have to first select $K(=2)$, independent treatments, let AB and AC will be confounded and hence ABAC = BC will also be confounded.

Now we will construct the intrablock subgroup as follows :

The treatment combination (1) should contained in the intra block. Now for other $2^{n-k} - 1 = 3$ treatment combinations which contained in the intra block subgroup will be such treatment combinations, each having an even number of letter in common with each of AB and BC, clearly ABC, ABCD and D (which is the generalised interaction of ABC and ABCD) are the three treatment combinations which have an even number of letters common (2, 2, 0 respectively) with AB and AC also with BC. Hence the intrablock is

(1)
abc
abcd
d

After obtaining the intra block subgroup, the other $2^k - 1$ blocks of the replicate are obtained one by one first including a treatment combination which has not appeared in the previous blocks constructed and then combining this letter with the letters of the treatment combinations of the intrablock subgroup and following the rule of rejecting a letter if it occurs twice.

For example, Consider 2^4 experiment in 2^2 blocks in a replicate. Suppose ABC and BCD (and hence AD) has to be confound.

Intra block	Block 2	Block 3	Block 4
(1)	a	b	d
bc	abc	c	bcd
abd	bd	ad	ab
acd	cd	abcd	ac

To construct Block 2, a is not appeared in intrablock and hence we put a in block 2.

Now, generalised intraction of a with the treatment combinations of intrablock are as follows:

$(bc)a = abc$, $(abd)a = bd$, $(acd)a = cd$.

and hence abc, bd and cd will be in block 2.

Note that, The intrablock is a normal subgroup of abelian group formed by small letters of treatment combination with composition as defined earlier for capital letters.

The other blocks are coset of that normal subgroup.

Again, the set of all small letters of treatment combination forms a vector space over Z_2 and the intrablock is a subspace of dimension $n-k$ and hence there are $n-k$ independent treatment in the intrablock and remaining $2^{n-k} - (n-k)$ can be constructed from this $n-k$ independent treatment combination using generalised interaction.

An example of 3^3-experiment in 3 blocks of 9 plot each

Spliting a replicate into 3 blocks of size 9 each will confounded 2 d.f. of treatment combinations in a replicate, This d.f. should be a component of 3-factor interaction.

There are $3^3 = 27$ treatment combinations which are $A^{C_1}B^{C_2}C^{C_3}$ where $C_1, C_2, C_3 \in \{0,1,2\}$.

Suppose, we decide to confound the AB^2C^2 and hence the generalised interaction of AB^2C^2 with itself which is A^2BC also confound with blocks. Now the intra block and the other two blocks are as follows:

Intra block	Block 2	block 3
1	c^2	b
ac	a	abc
ab	abc^2	ab^2
a^2c^2	a^2c	a^2bc^2
a^2b^2	$a^2b^2c^2$	a^2
a^2bc	a^2c	a^2b^2c
bc^2	bc	b^2c^2
cb^2	b^2	c
ab^2c^2	ab^2c	ac^2

Alternating way to form blocks in a confounding factorial design :

If we want to construct the block by the tecrique which describe above computation is complicated. For this reason we introduce a different method to construct the blocks. The method is based on solving the linear equations.

System of linear Equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_n$$

The above system of linear equation is of n variables and m equations.

If all b_i's are 0, then the system is called system of homogenous linear equations. It has a trivial solution $x_1 = 0, x_2 = 0, \cdots, x_n = 0$.

If $(\alpha_1, \alpha_2, \cdots \alpha_n)$ and $(\beta_1, \beta_2, \cdots, \beta_n)$ are two solution then $(\alpha_1+\beta_1, \alpha_2+\beta_2, \cdots, \alpha_n+\beta_n)$ and $(k\alpha_1, k\alpha_2, \cdots, k\alpha_n)$ are also solution for any real k. Hence set of all solution of a system of homogenous linear equations form a vector subspace of $\mathbb{R}^n$.

We have the following result

number of independent equations + dimension of the solution set = number of variables.

2^3 factorial design

Consider 2^3 factorial design, we represent the treatment combinations as 000, 001, 010, 011, 100, 101, 110, 111. In place of (1), c, b, bc, a, ac, ab, and abc respectively. The generalised interactions of 2 treatment combinations can be found Xoring the 2 treatment (i.e. adding under modulo 2), as for example the generalised interaction of 101 and 011 is 110 as

$$\begin{array}{r} 101 \\ 011 \\ \hline 110 \end{array}$$

Consider the following equations.

$$\left.\begin{aligned} 1.x_1 + 0.x_2 + 0.x_3 &= 0 \\ 1.x_1 + 0x_2 + 0x_3 &= 1 \end{aligned}\right\} \quad \cdots (1)$$

$$\left.\begin{aligned} 0.x_1 + 1.x_2 + 0.x_3 &= 0 \\ 0.x_1 + 1.x_2 + 0.x_3 &= 1 \end{aligned}\right\} \quad \cdots (2)$$

$$\left.\begin{aligned} 0.x_1 + 0.x_2 + 1.x_3 &= 0 \\ 0.x_1 + 0.x_2 + 1.x_3 &= 1 \end{aligned}\right\} \quad \cdots (3)$$

$$\left.\begin{aligned} 1.x_1 + 1.x_2 + 0.x_3 &= 0 \\ 1.x_1 + 1.x_2 + 0x_3 &= 1 \end{aligned}\right\} \quad \cdots (4)$$

$$\left.\begin{aligned} 1.x_1 + 0.x_2 + 1.x_3 &= 0 \\ 1.x_1 + 0.x_2 + 1.x_3 &= 1 \end{aligned}\right\} \quad \cdots (5)$$

$$\left.\begin{aligned} 0.x_1 + 1.x_2 + 1.x_3 &= 0 \\ 0.x_1 + 1.x_2 + 1.x_3 &= 1 \end{aligned}\right\} \quad \cdots (6)$$

$$\left.\begin{aligned} 1.x_1 + 1.x_2 + 1.x_3 &= 0 \\ 1.x_1 + 1.x_2 + 1.x_3 &= 1 \end{aligned}\right\} \quad \cdots (7)$$

The solutions of each of the equations divides the 8 treatment combinations into 2 subsets each contained 4 solutions. Comparison between these 2 subset totals gives different treatment effects depending on the system of linear equations from which the solutions are obtained.

For example Consider the equation (2), the set of solutions of $0.x_1 + 1x_2 + 0x_3 = 0$ are $\{000, 100, 001, 101\}$, and the set of solutions of $0x_1 + 1.x_2 + 0x_3 = 1$ are $\{011, 110, 010, 111\}$

This two subsets gives a partition of the set of all treatments comparison between these two subset total gives the main effect ~~of~~ B. If we want to confound the main effect B with the blocks, then we apply this two subsets in two different blocks.

Again, consider the equation (4), the set of solutions of $1.x_1 + 1x_2 + 0.x_3 = 0$ is $\{000, 110, 001, 111\}$ and the set of solutions of $1.x_1 + 1.x_2 + 0x_3 = 1$ is $\{101, 010, 011, 100\}$.

These two subset gives a partition of the set of all treatments. Comparison between these two subset totals gives the ~~necessi~~ effect AB. If we wan to confound AB with blocks, then we apply this two subsets in two different blocks.

2^n-factorial Design

In general, for 2^n factorial design the treatment combinations are represented as n-tuple $x_1 x_2 \ldots x_n$, where $x_i \in \{0,1\}$, $\forall\ i = 1(1)n$.

As an example the treatment (1) is denoted by $000\ldots0$, the treatment A_2 is denoted by $010\ldots0$ and the treatment $A_1A_2A_4$ is denoted by $11010\ldots0$. The generalised interaction of two treatment combination can be obtained adding these two treatments under modulo 2.

There are $2^n - 1$ system of linear equations, each system contained two linear equations, one homogeneous and one non-homogeneous.

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$$

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = 1$$

where $a_i \in \{0,1\}$, $i = 1(1)n$.

Different values of a_i's gives different system of equations.

Using vector notation a treatment combination $(x_1, x_2, \ldots x_n)$ is an element of $Z_2^n = Z_2 \times Z_2 \times \ldots \times Z_2$, and for two treatment combinations $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n) \in Z_n$ and $\underset{\sim}{y} = (y_1, y_2, \ldots y_n) \in Z_n$, the generalised interaction is the ~~bitwise addi~~ coordinatewise adding of $\underset{\sim}{x}$ and $\underset{\sim}{y}$ ie.

$$\underset{\sim}{x} + \underset{\sim}{y} = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$$

For a fix $\underset{\sim}{a} = (a_1, a_2, \ldots, a_n) \neq \underset{\sim}{0} = (0, 0, \ldots, 0)$

$\underset{\sim}{a} \cdot \underset{\sim}{x} = 0$ and $\underset{\sim}{a} \cdot \underset{\sim}{x} = 1$

gives two of linear equations, one of which is homogeneous and other one is non-homogeneous.

Since, there are 2^n elements in $\mathbb{Z}_2^n$ there are 2^n treatment combinations and $2^n - 1$ system of equations because $\underset{\sim}{a} \neq \underset{\sim}{0}$.

Now $\mathbb{Z}_2^n$ is a vector space of dimension n over the field $\mathbb{Z}_2$. The set of solutions of homogeneous equations $\underset{\sim}{a} \cdot \underset{\sim}{x} = 0$ i.e. $\sum_{i=1}^{n} a_i x_i = 0$ forms a vector subspace of dimension $(n-1)$.

Now to find the set of all solutions of the above homogeneous equation, we have to find $n-1$ independent solutions and remaining solutions are linear combination of these $(n-1)$ independent solutions.

To find the solution of the non-homogeneous equations $\underset{\sim}{a} \cdot \underset{\sim}{x} = 1$

we will first find a particular solution say $\underset{\sim}{x}_1 \in \mathbb{Z}_2^n$, then the set of solutions of the above equation is $\{\underset{\sim}{x}_1 + \underset{\sim}{x}_0 : \underset{\sim}{a}_0 \cdot \underset{\sim}{x}_0 = 0, x_0 \in \mathbb{Z}_2^n\}$.

Note that, $\underset{\sim}{x}_0$ is a solution of homogeneous equation $\underset{\sim}{a} \cdot \underset{\sim}{x}_0 = 0$ and $\underset{\sim}{x}_1$ is a solution of $\underset{\sim}{a} \cdot \underset{\sim}{x} = 1$ then $\underset{\sim}{x}_0 + \underset{\sim}{x}_1$ is a solution of $\underset{\sim}{a}\underset{\sim}{x} = 1$.

$\underset{\sim}{x}_0 + \underset{\sim}{x}_1$ is the treatment combinations which is generalized interaction of $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_0$.

The set of solutions of each of the two equations

$\underset{\sim}{a}.\underset{\sim}{x} = 0$ and $\underset{\sim}{a}.\underset{\sim}{x} = 1$

divides the 2^n treatment into two subsets. These t
subsets gives a partition of set of all treatments
Comparison between these two subsets total gives the
effects of the treatment $A_1^{a_1} A_2^{a_2} \cdots A_n^{a_n}$,
where $\underset{\sim}{a} = (a_1, a_2, \cdots, a_n)$. So this effect can be
represented by $\underset{\sim}{a}$ itself.

Now if we want to confound the treatment
$\underset{\sim}{a} = (a_1, a_2, \cdots a_n)$ the we apply the above two
subsets of solutions in two different blocks.

A 3^3 experiment in blocks of 9 plot each.

There one 27 treatment $A^{x_1} B^{x_2} C^{x_3}$ where
$x_1, x_2, x_3 \in \{0,1,2\}$. The treatment $A^{x_1} B^{x_2} C^{x_3}$ will
be denoted as (x_1, x_2, x_3) or simply $x_1 x_2 x_3$. As for
example AB and AB^2C will be denoted by
$(1,1,0)$ and $(1,2,1)$ or simply by 110 and
121 respectively. Splitting a replicate into 3
blocks of size 9 each will confound 2 d.f.
of treatment. This d.f. should be a
component of 3 factor interaction.

Suppose, we decide to confound the AB^2C^2
component with blocks in a replicate. The
defining equations for AB^2C^2 are

$$x_1 + 2x_2 + 2x_3 = i \pmod 3$$

$i = 0, 1, 2.$

The set of solutions of these three equations partition the 27 treatments into 3 subsets, these 3 subsets will be applied to 3 blocks of a replicate at random and inside a block the 9 treatments will be further randomised.

$x_1 + 2x_2 + 2x_3 = 0$

000
110
101
211
220
202
122
021
012

$x_1 + 2x_2 + 2x_3 = 1$

100
222
020
002
011
112
201
210
121

$x_1 + 2x_2 + 2x_3 = 2$

200
111
001
022
010
221
102
120
212

Complete and Partial Confounding

When there are two or more replications a question arises whether the same interaction are confounded in each replication or different sets of interactions are confounded in different replications. If the same set of interactions is confounded in all the replications confounding is called complete. If different sets of interactions are confounded in different replications confounding is called partial.

Example We can get the two groups of eight treatments each for 2^4 factorial from the solutions of the equations :

$$\left.\begin{aligned} x_1+x_2+x_3+x_4 &= 0 \\ x_1+x_2+x_3+x_4 &= 1 \end{aligned}\right\}$$

The interaction ABCD is accordingly confounded here If partial confounding is desired we can confound another interaction, say ABC in the second replicatio The blocks are then obtained from the solutions of the equations :

$$\left.\begin{aligned} x_1+x_2+x_3 &= 0 \\ x_1+x_2+x_3 &= 1 \end{aligned}\right\}$$

We notice that one of the two equations for getting the two blocks contents is homogeneous. So, all linear combinations of the solutions vectors of the homogeneous equations are also its solutions. By obtaining three independent solutions, all the other solution can be obtained from their linear combinations. In general, if the size of the block corresponding to a set of homogeneous equations is 2^r, its contents can be generated from all possible linear combinations of r independent solutions of the equations. Such a block is called principal or key block.

Again, if $(s_1, s_2, \ldots, s_m, \ldots s_n)$ is a solution of the equation $\sum_{i=1}^{n} p_i x_i = 0$ then $(s_1, s_2, \ldots, s_m+1, \ldots s_n)$ is a solution of the equation $\sum p_i x_i = 1$ if $p_m \neq 0$ i.e. $p_m = 1$.

Hence to get the second block from the contents of the key block, that is, by increasing mth element by 1 in all the treatments in the key block, we get the other.

To confound, more than one interaction, the key block content are obtained from the solutions of more than one homogeneous equation simultaneously.

For example; we get the key block of size 2^3 in 2^5 experiment from the solutions of the equations : $x_1 + x_2 + x_3 = 0$

and $x_1 + x_4 + x_5 = 0$

This equation indicate that ABC and ADE interactions are confounded simultaneously in the same replication. Now any solution of the above two homogeneous equations is also a solution of the equation which is obtained from a linear combination of the two equations.

This shows $x_2 + x_3 + x_4 + x_5 = 0$ and no other equation is possible. Hence the interaction BCDE is also confounded and no other is confounded. This interaction is generalised interaction of the previous two interaction.

The key block has been obtained as shown blow by first obtaining three independent solutions of the homogeneous equations and then taking all their linear combinations.

Hence the key block is

0	0	0	0	0	
0	1	1	1	1	three independent solution.
1	0	1	0	1	
1	0	1	1	0	
1	1	0	1	0	
1	1	0	0	1	
0	0	0	1	1	
0	1	1	0	0	

The other three blocks are obtained from the solutions of the following three set of equations

$$x_1 + x_2 + x_3 = 1 \quad ; \quad 0 \quad ; \quad 1$$

$$x_1 + x_4 + x_5 = 0 \quad ; \quad 1 \quad ; \quad 1$$

The solution can obtained just by finding only one solution to each sets and adding the solution vector to all the treatment vectors in the key block.

We have seen that when there are four blocks in a replication, three interactions are confounded, two of which are independent and one generalized. When there are 2^k blocks in a replication, $2^k - 1$ interactions are confounded of which k are independent and the remaining $2^k - k - 1$ are generalised. We have enough control over the choice of independent interactions but very little control on the generalised interaction. This fact creates difficulty in the construction so as to save all lower order interaction including main effect

Confounding in more than 2 blocks of an 2^n factorial design

Let us consider 2^n factorial design in 2^k blocks each contained 2^{n-k} plots per replicate, where n and k are two natural numbers. Here we have to confound k independent treatments $\underset{\sim}{a_1} = (a_{11}, a_{12}, \cdots a_{1n}) \in \mathbb{Z}_2^n$

$\underset{\sim}{a_2} = (a_{21}, a_{22}, \cdots a_{2n}) \in \mathbb{Z}_2^n$, $\cdots\cdots$, $\underset{\sim}{a_k} = (a_{k1}, a_{k2}, \cdots a_{kn}) \in \mathbb{Z}_2^n$

Then any linear combinations, other than (1), of these k independent treatments (i.e. generalised interaction) also confounded; i.e. any treatment in the linear span $L(\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \cdots \underset{\sim}{a_k}\})$ of $\{\underset{\sim}{a_1}, \underset{\sim}{a_2}, \cdots \underset{\sim}{a_k}\}$ other than $(0,0,\cdots 0)$ also confounded.

To determine 2^k blocks, we consider 2^k system of linear equations, each system contains k linear equations as follows:

$$\underset{\sim}{a_1}\,\underset{\sim}{x} = 0 \text{ or } 1$$

$$\underset{\sim}{a_2}\cdot\underset{\sim}{x} = 0 \text{ or } 1 \qquad \underset{\sim}{x} = (x_1, x_2, \cdots x_n) \in \mathbb{Z}_2^n$$

$$\vdots$$

$$\underset{\sim}{a_k}\,\underset{\sim}{x} = 0 \text{ or } 1$$

Among these 2^k systems, one system is homogeneous and the others are non-homogeneous. Any system of linear equations can be represent as follows.

$$A\,\underset{\sim}{x}^T = \underset{\sim}{b}^T$$

where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{pmatrix}$ $\underset{\sim}{x} = (x_1, x_2, \cdots x_n)$

$\underset{\sim}{b} = (b_1, b_2, \cdots, b_k)$

$$A = \begin{pmatrix} \underset{\sim}{a_1} \\ \underset{\sim}{a_2} \\ \vdots \\ \underset{\sim}{a_k} \end{pmatrix} \in M_{k\times n}(\mathbb{Z}_2), \quad \underset{\sim}{b} \in \mathbb{Z}_2^k, \quad \underset{\sim}{x} \in \mathbb{Z}_2^n.$$

For each $\underset{\sim}{b}$, the solution set of the system of equations gives a block.

In particular when $\underset{\sim}{b} = 0$, system of linear equation is homogeneous, the block is known as initial block, the solution set is a subspace of dimension k. So we will find k independent solution and the construct initial block.

Now to construct a block whose defining equation is $A\underset{\sim}{x}^T = \underset{\sim}{b}^T \quad (\underset{\sim}{b} \neq 0)$, we first find a solution of the above equation. Then we compose (componentwise addition under modulo 2) this particular solution to every solution of the system $A\underset{\sim}{x}^T = \underset{\sim}{0}$ and this composition will give all solutions of $A\underset{\sim}{x}^T = \underset{\sim}{b}^T$ and hence the block.

Note; if $\underset{\sim}{x}_0$ is a solution of $A\underset{\sim}{x}^T = \underset{\sim}{0}$ and x_1, x_2 two solutions of $A\underset{\sim}{x}^T = \underset{\sim}{b} \ (\underset{\sim}{b} \neq \underset{\sim}{0})$ then $\underset{\sim}{x}_0 + \underset{\sim}{x}_1$ is a solution of $A\underset{\sim}{x}^T = \underset{\sim}{b}$ and $\underset{\sim}{x}_2 - \underset{\sim}{x}_1$ is a solution of $A\underset{\sim}{x}^T = \underset{\sim}{0}$.

Experiments with Factors at three level each

When factors are taken at three levels instead of two, the scope of the experiments increase. Further, when the level of a factor are quantitative, the pattern of change of response with the increase of the level values of the factor can be studied in a better way. A study to investigate if the change is linear or quadratic is possible when the factors are at three levels.

Let us consider two factor A and B each at three levels. We get the following treatment combinations or simply treatments 00, 01, 02, 10, 11, 12, 20, 21, 22.

A randomised block design can be adopted for the experiment. The s.s. can be obtained either from the total of the s.s. due to the two contrasts or form the three group totals by the usual method of deviation squares. Let A_0, A_1 and A_2 denote the totals of the three groups of treatments obtained. Then the contrast $A_2 - A_0$ give the linear contrast among the three levels of A if they are equispaced. The other contrast orthogonal to the above is $A_2 - 2A_1 + A_0$.

This contrast indicates a quadratic relation between the levels and their responses. The main effect of B can also be treated similarly.

The interaction s.s. can be obtained by subtracting s.s. due to A and B from the treatment ss when the effect of B is different at different levels of A, the factors are said to have interaction.

The interaction can be split into components of interactions between the linear and quadratic effects of the two factors, the four interaction components can be written as A_L, B_L, $A_L B_Q$, $A_Q B_L$ and $A_Q B_Q$. They can be obtained from the following contrasts among the levels of the two factors:

$A_L B_L = (a_2 - a_0)(b_2 - b_0) = a_2 b_2 - a_2 b_0 - a_0 b_2 + a_0 b_0$

$A_L B_Q = (a_2 - a_0)(b_2 - 2b_1 + b_0)$ etc.

Similarly the other contrasts.

Though this way of subdivision of the interaction effect is more informative, such components cannot be used for the construction of 3^n factorials.

Let us define, one variate for each factor in a factorial such that ith variate x_i denotes the levels of the ith factor in the different combinations of the factors. In the 3^2 factorial we have two variates x_1 and x_2 corresponding to A and B respectively.

Using the solutions of the equations:

$x_1 = 0$

$x_1 = 1$

$x_1 = 2$

we can divide the nine treatments into three non-overlapping groups of three each.

These three groups are obtained for finding the main effects of A.

The interaction component obtained from the equations $x_1 + x_2 = i \quad (i = 0, 1, 2)$ is denoted by AB.

Similarly for B, AB^2.

Note that AB and A^2B^2 are same, so also AB^2 and A^2B.

Factorials with four factors each at three levels

When there are four factors we have to get nine blocks of nine plots each. The 3^4 combinations have to be made into 3^2 groups. We have to use two homogeneous equations simultaneously to get contents of the key block.

For example, we take the following two equations:

$$\left.\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + 2x_2 + x_4 &= 0 \end{aligned}\right\}$$

to confounding ABC and AB^2D.

The solutions of these equations give the key block contents. As these are homogeneous equations, any solution is also is a solution of an equation obtained from their linear combinations. Hence two more interactions corresponding to the following two resulting equations are also confounded:

$$x_1 + 2x_3 + 2x_4 = 0$$

$$x_2 + 2x_3 + x_4 = 0$$

Therefore, the interactions AC^2D^2 and BC^2D are also confounded. These two interactions are the generalised interactions. The other blocks are obtained from the solutions of the following eight equations.

$x_1+x_2+x_3 =$	0	0	1	1	1	2	2	2
$x_1+2x_2+x_4 =$	1	2	0	1	2	0	1	2
Block	2	3	4	5	6	7	8	9

The key block is

0	0	0	0
1	1	1	1
2	0	1	1
0	1	2	1
2	2	2	0
1	0	2	2
0	2	1	2
1	2	0	1
2	1	0	2

Other eight blocks can also be obtained from the key block by an exactly similar procedure as described in 2^n factorials. Here, again, by obtaining only two independent solutions of the homogeneous equations we can get the others from their linear combinations.

In general, if in 3^n factorial block size 3^r, the key block can be obtained from the linear combinations of r independent solutions of $n-r$ homogeneous equations corresponding to confounded interactions. Again out of the $\frac{3^{n-r}-1}{2}$ confounded interactions, $n-r$ are independent and rest are generalised interaction.

A general method of Construction of Confounded Factorials:

Let there be n factors each at s levels where $s = p^m$ and p is a prime. The total number of treatments in the factorial is s^n. There are $(s^n - 1)$ d.f, due to the treatments. These are divided into $\frac{s^n - 1}{s-1}$ main effect and interaction components each of $s-1$ d.f. There are $(s-1)^{k-1}$ components belongs to interaction among a given set of k factors, becaus if

~~The components can be represented by equations of the following form:~~

$T_1, T_2, \ldots, T_k$ are k factors., Then the interactions of these k factors are of the form

$$T_1^{s_1} T_2^{s_2} \ldots T_k^{s_k} \quad \text{where } s_1, s_2, \ldots s_k \in \{0, 1, \alpha_2, \ldots, \alpha_{s-1}\}$$

The two treatment combinations $T_1^{s_1} T_2^{s_2} \ldots T_k^{s_k}$ and $T_1^{s_1'} T_2^{s_2'} \ldots T_k^{s_k'}$ are same. If $s_i' = \lambda s_i \ \forall \ i = 1, 2, \ldots k$ and for some fixed $\lambda \in \{1, \alpha_2, \ldots \alpha_{s-1}\}$.

The components $T_1^{p_1} T_2^{p_2} \ldots T_n^{p_n}$ can be represented by equation of the following form:

$$p_1 x_1 + p_2 x_2 + \ldots + p_n x_n = i \quad (i = 0, 1, \alpha_2, \alpha_3, \ldots, \alpha_{s-1})$$

where $p_1, p_2, \ldots p_n \in GF(p^m) = \{0, 1, \alpha_2, \alpha_3, \ldots \alpha_{s-1}\}$

$0, 1, \alpha_2, \alpha_3, \ldots, \alpha_{s-1}$ are distinct elements of $GF(p^m)$ and $x_1, x_2, \ldots x_n$ are variate corresponding to the factor discussed as earlier.

Each choice of the p_i's (not all zero) gives the equation of a component. Once a set of p_i's has been selected, all its multiples are omitted as they donot lead to any new grouping.

A set of k non-zero p_i's in the equation lead to an interaction component of those k factors, the variates of which occur with non-zero coefficients in the equation, If in an equation the variate x_i has the non-zero coefficient p_i, the in the interaction corresponding to it, the corresponding factor A_i has the power p_i. It can be shown that the group contrasts for any interaction compohent is orthogonal to the groups contrasts of any other interaction.

In the confounded factorial s^n let the block size be s^r. There are s^{n-r} blocks. So $\frac{s^{n-r}-1}{s-1}$ interaction components are confounded. In the key block only r treatments are independent. The other can be obtained from their linear combinations. Among the $\frac{s^{n-r}-1}{s-1}$ confounded interactions $(n-r)$ are independent and the rest are generalized. The contents of the different blocks can be obtained from the solution of a set of $n-r$ equations following from the independent confounded interactions.

An alternative Method

In the previous method (due to Bose) the interactions for confounding are chosen first and the block contents follow.

In this method (due to Das) the block contents are chosen first and the interactions confounded follow from there.

When the block size is s^r, we begin with the following r independent treatment combinations of r factors each at s levels, written in the form of an $r \times r$ identity matrix.

A_1	A_2	A_3	. .	A_r	A_{r+1}
1	0	0	- -	0	0
0	1	0	- -	0	0
.	.	-	-		.
.		-	-		.
0	0	0	- - -	1	0

All the s^r treatment combinations of the r factors can be generated from the linear combinations of these r treatments.

Next, to accommodate more factors we go on annexing further columns, one by one, to the right of the above matrix. Each such column corresponds to a subsequently taken factor. They are denoted by $A_{r+1}, A_{r+2}, - - -, A_n$.

Let us now decide the column below A_{r+1}. The columns evidently consist of the elements of the field $GF(p^m)$. As soon as a column below A_{r+1}

is written, the r combinations each of r+1 factors, become the r independent combinations of the key block. Now if the column consists of zero only, all the elements under the factor A_{r+1} will be zero when the r independent combinations are linearly combined to get the complete key ~~block~~block. Hence in this case the main effect of A_{r+1} is confounded, because the elements of the key block satisfying the equation $x_{r+1} = 0$.

So to save main effects a column consisting of zeros only should not be taken.

Next, let us take a column which is a multiple of one of the columns already present. Here, multiplication is in the sense of multiplying a vector by a scalar. Let x_{r+1} which denotes the variates for the factor A_{r+1}, be p_i times an already existing ith column which corresponds to the factor A_i and variate x_i.

Then $x_{r+1} = p_i x_i \qquad p_i \in GF(p^m)$

i.e. $x_{r+1} + (p-1)p_i x_i = 0 \quad (\because -p_i = p-1)$

This shows that a two factor interaction component involving the factor A_{r+1} and A_i is confounded $\left(A_{r+1} A_i^{(p-1)p_i} \text{ confounded} \right)$.

Also The equation can be written also ~~like~~ as

$(p-1) x_{r+1} + p_i x_i = 0$

Then $A_{r+1}^{p-1} A_i^{p_i}$ is confounded, note that $A_{r+1}^{p-1} A_i^{p_i}$ and $A_{r+1} A_i^{(p-1)p_i}$ are same.

Hence to save the two factor interaction components no column should be taken any of whose multiples or is already existing. This should maintain for selecting any subsequent column.

Similarly, to save three factor interactions no column should be taken which is a linear combinations of any two existing column. This concept extends for saving interaction of any order.

After a column below A_{r+1} has been chosen, another column below A_{r+2} is chosen similarly. This procedure continues till the last column.

For each subsequent factor taken, an independent interaction is confounded. Let the column taken below A_{r+1} be the vector $(\alpha_1, \alpha_2, \ldots \alpha_r)^T$. Denote the column vectors below the first r factors by $\underset{\sim}{e}_1^T, \underset{\sim}{e}_2^T, \ldots, \underset{\sim}{e}_r^T$, these are unit vectors.

Hence $x_{r+1} = \alpha_1 \underset{\sim}{e}_1 + \alpha_2 \underset{\sim}{e}_2 + \cdots + \alpha_r \underset{\sim}{e}_r$

i.e. $\alpha_1 \underset{\sim}{e}_1 + \alpha_2 \underset{\sim}{e}_2 + \cdots + \alpha_r \underset{\sim}{e}_r^T + (p-1)\, x_{r+1} = \underset{\sim}{0}$

Hence this equation

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_r x_r + (p-1) x_{r+1} = 0$$

satisfied by $x_1 = \underset{\sim}{e}_1^T$, $x_2 = \underset{\sim}{e}_2^T$, $\ldots$ $x_r = \underset{\sim}{e}_r^T$ and $x_{r+1} = (\alpha_1, \alpha_2, \ldots \alpha_r)^T$.

This equation gives the interaction confounded as a result of choosing the column $(\alpha_1, \alpha_2, \ldots \alpha_r)^T$ below A_{r+1}.

Therefore, the interaction is $A_1^{\alpha_1} A_2^{\alpha_2} \ldots A_r^{\alpha_r} A_{r+1}^{(p-1)}$

When the power of a factor is zero, it goes out of the interaction. The interaction confounded due to the choice of a column is independent of the interaction corresponding to any previously chosen chosen column, as every time a new column is taken.

In this way by choosing $(n-r)$ columns according to requirement, we get the key block for the confounded factorial together with the independent interactions confounded.

The generalised interactions can be obtained from the possible linear combinations of the chosen columns below A_{r+1} to A_n and then using the resulting column just like the chosen column for writing the generalised interaction.

The heading of a thus derived column is not a single letter but the product of those factor with appropriate powers whose column have been combined. If a particular column variate occurs with a coefficient p_t in one of the column combinations the factor for this column has the power p_t in the generalised interaction.

$4^3 \times 4^2 = 4^5$

$1+\alpha=0$ $\quad \alpha^2+\alpha=0$ $\quad 1+\alpha+\alpha^2=0$

Example Construction of 4^5 factorial in 4^2 plot block saving all main effects and two factor interaction components.

Key block

A_1	A_2	A_3	A_4	A_5	Some derived column for generalized interactions		
x_1	x_2	x_3	x_4	x_5	x_3+x_4	$x_3+\alpha x_4$	$x_3+\alpha^2 x_4$
1	0	1	1	1	0	α^2	α
0	1	α	α^2	1	1	α^2	0

Where $GF(2^2) = \{0, 1, \alpha, \alpha^2\}$

$1+0+1=0$
$0+\alpha^2+\alpha^2=0$

The last two rows gives the two independent treatments of the five factors in the key block. The interaction confounded due to column below A_3 is: $A_1 A_2^{\alpha} A_3$. as $x_3 = x_1 + \alpha x_2$

i.e. $x_1 + \alpha x_2 + x_3 = 0$

Similarly, the interaction confounded for other two columns under A_4 and A_5 are $A_1 A_2^{\alpha^2} A_4$, $A_1 A_2 A_5$.

The generalized interactions for three derived columns shown in the table are $A_2 A_3 A_4$, $A_1^{\alpha^2} A_2^{\alpha^2} A_3 A_4^{\alpha}$ and $A_1^{\alpha} A_3 A_4^{\alpha^2}$.

The next 15 generalised interactions can be obtained from the other linear combinations of the last three columns. key block can be construct from the linear combination of two treatment 10111, 01 $\alpha\alpha^2$ 1. The other 63 blocks can be obtained by taking a treatment which has not occured in the key-block and adding this treatment to each of the treatments in the key block.

Balanced Designs

If in a partially confounded design each of the interaction components of a particular order is confounded an equal number of times which may be different for different orders of interaction, then the design is called balanced for such interactions.

Case I The 3^3 design in block of size 3^2 when obtained in 4 replications by confounding the different replications the four possible 3-factor interaction components, ABC, ABC^2, AB^2C, and AB^2C^2, is a balanced design for three factor interaction components. (AB^2C^2 → A^2BC)

Case II The 2^5 factorial can have a confounded design in 2^3 plot blocks. The different key blocks of different replications can be obtained as follows:

The following independent treatment combinations of three factors A, B and C are first taken:

A	B	C
1	0	0
0	1	0
0	0	1

D E

Next from the following scheme:

1	2	3	4
1	1	0	1
1	0	1	1
0	1	1	1

We choose for factors D and E two consecutive columns in all possible ways. By annexing each such pair with column under A, B and C we get the key blocks of 4 replicates.

The independent treatment combinations of these 4 replication are the following:

Rep I

A	B	C	D	E
1	0	0	1	1
0	1	0	1	0
0	0	1	0	1

(1 and 2)

Rep II

A	B	C	D	E
1	0	0	1	0
0	1	0	0	1
0	0	1	1	1

(2 and 3)

Rep III

A	B	C	D	E
1	0	0	0	1
0	1	0	1	1
0	0	1	1	1

(3 and 4)

Rep IV

A	B	C	D	E
1	0	0	1	1
0	1	0	1	1
0	0	1	1	0

(4 and 1)

In these replications the following interactions are confounded

ABD	ACD	BCD	ABCD
ACE	BCE	ABCE	ABE
BCDE	ABDE	ADE	CDE

The following interactions are not present

- ABC
- BDE
- ACDE

The key blocks corresponding to these interac-

The key block corresponding to this interaction confounded is

A	B	C	D	E
1	0	1	0	0
0	1	1	1	0
0	0	0	1	1

Thus we get a balanced design with the above five replication's.

Case 3

A balanced design for 3^4 factorial in 3^2 plot blocks.

Start with the following combinations of the factors A and B and the two more column for the two other factors C and D are to be annexed

A	B	C	D
1	0		
0	1		

The columns for C, D are to be chosen consecutively from the following 4 columns

1	2	2	1
1	1	2	2

The independent treatment combinations in the four replications obtained as follows.

A	B	C	D
1	0	1	2
0	1	1	1

A	B	C	D
1	0	2	2
0	1	1	2

A	B	C	D
1	0	2	1
0	1	2	2

A	B	C	D
1	0	1	1
0	1	2	1

The interaction confounded are :

$A B C^2$	$A B^2 C$	$A B C$	$A B^2 C^2$
$A B^2 D$	$A B C$	$A B^2 D^2$	$A B D^2$
$A C D^2$	$A C^2 D^2$	$A C^2 D$	$A C D$
$B C D$	$B C D^2$	$B C^2 D^2$	$B C^2 D$

It will be seen that each of the possible 16, three factor interaction components is confounded once in the above 4 replications. Thus the design is balanced.

Maximum Number of factors to save interactions up to a given order for given Block size:

When block size is given it is of interest to ascertain the maximum number of factors that can be accommodated in the given block size so as to save all interactions up to a given order.

Let the block size be s^r. To obtain confounded interactions and the key block we take an $r \times r$ unit matrix and annex a further column to it.

In order to save all main effects and two factor interactions we cannot take any multiple of a column already taken, nor the null column. There are in all $s^r - 1$ possible columns including the r columns in the unit matrix but excluding the null column. If we take one of these column, we are to exclude all its $(s-2)$ multiples to save two factor interactions. Thus, in a group of $s-1$ columns which are mutual multiples, only one column can be taken.

Hence the maximum number of column that can be taken is $\frac{s^r-1}{s-1}$,

Therefore, the maximum number of factors each of s levels that can be accommodated in a block of size s^r without confounding any main effect and two factor interactions is $\frac{s^r-1}{s-1}$.

Example When the block size is 2^3. We can accommodate seven factors to save all main effects and two factor interactions. The independent treatments in the key block are as follows.

A	B	C	D	E	F	G
1	0	0	1	1	0	1
0	1	0	1	0	1	1
0	0	1	0	1	1	1

In the example of 4^5 factorial in 4^2 plot blocks, five is the maximum number of factors for 4^2 plot blocks to save all main effects and two factor interactions. Therfore, it is not possible to annex any further column to the key block presented before for saving main effects and two factor interactions.

Finite Fields

Construction and analysis of several types of designs becomes very convenient by using the properties of finite fields. We have discussed below some of these properties which we shall require.

1. For an positive integer p, $a \equiv b \pmod p$ if $a-b$ is divisible by p i.e. a and b left same remainder when they divided by p.

2. Addition, subtraction, multiplication of two elements under modulo p is possible. Division is possible for some cases and if p is a prime then division is possible after all non zero elements.

For example, when $p = 7$, the elements of mod p are $0, 1, 2, 3, 4, 5, 6$. Here $3+4 = 0$, $3-4 = 6$, $3 \times 4 = 5$, $3 \div 4 = 6$.

3. When any element of a prime module is multiplied in turn by all of its non zero elements, each time a different product is obtained. This ensures all possible divisions by non zero elements. In this case the elements forms a finite field of order p, which is known as Galois field and written as $GF(p)$.

There is at least one element in every finite field, different powers of which give the different non-zero elements of the field. Such an element is called primitive root of the field.

When $p = 7$, the primitive root is 3, because:

$3^0 = 1$, $3^1 = 3$, $3^2 = 2$, $3^3 = 6$, $3^4 = 4$, $3^5 = 5$, $3^6 = 1$

Finite field or Galois field

Let the number of elements be $S = p^n$ where p is a prime and n any positive integer. Let us now define s elements consisting of 0 and $s-1$ polynomials up to $(n-1)$ degree written by using any symbol, say, α. The coefficients in the polynomials are the elements of mod p. With these elements as coefficients the total number of such polynomials is $s-1$.

For example, 3^2 polynomial up to degree one are written below using the elements of mod 3. as coefficients:

$0, 1, 2, \alpha, \alpha+1, \alpha+2, 2\alpha, 2\alpha+1, 2\alpha+2$.

These elements can be added and subtracted mod p, but they cannot be multiplied or divided unless a device for reducing the polynomials is obtained. Such a device consists of choosing an irreducible polynomial of degree n and then any polynomial of degree n or more is equated to the remainder when it is divided by the irreducible polynomial. Such reducing polynomials are called minimal polynomials, It is not always easy to obtain them. But some, which are required often are given below:

Field	minimal polynomials
2^2	$\alpha^2 + \alpha + 1$
2^3	$\alpha^3 + \alpha + 1$
3^2	$\alpha^2 + \alpha + 1$

There may be more than one minimal polynomial in a field.

The non zero elements can be obtained from different powers of α up to α^{p^n-2}.

For example, in the 3^2 field with minimal polynomial $\alpha^2 + \alpha + 2$

$\alpha^0 = 1$, $\alpha^1 = \alpha$, $\alpha^2 = 2\alpha + 1$, $\alpha^3 = 2\alpha + 2$

$\alpha^4 = 2$, $\alpha^5 = 2\alpha$, $\alpha^6 = \alpha + 2$ and $\alpha^7 = \alpha + 1$

α^{p^n-1} is always 1, Here $\alpha^8 = 1$.

Multiplication and division became very easy by using such powers as elements.

$(\alpha+2)(2\alpha+2) = \alpha^6 \cdot \alpha^3 = \alpha^9 = \alpha$

$(2\alpha+2) \div (\alpha+2) = \alpha^3 \div \alpha^6 = \alpha^{10} \div \alpha^6 = \alpha^4 = 2\alpha$

A	B	C	D	E	F	G
1	0	0	1	0	1	1
0	1	0	1	0	1	0
0	0	1	0	1	0	1

*(BIBD)
(MN SIR)

Hence, the confounded treatments are

~~AD E G, BD E F, C E F G.~~

ABD, BCE, ABCF, ACG

and there generalized ~~Con~~ interactions.

~~Hence, the confounded treatments are~~

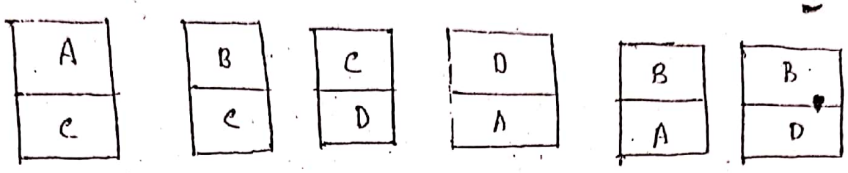
e.g.2:- 4⁵ factorial in 4² plot ~~block~~ size :-

So, 5 is max. no. of factors to save all the main effects and 2 factors interaction. It is not possible to annexed to any further column to the key block presented before to for saving main effect and 2 factor interactions

12.09.2

Incomplete block design (IBD):- An IBD is a design having v treatments b blocks each of size k such that each of the treatment is replicated r times and $v > k$. v, b, r, k are known as parameters of IBD.

e.g.:- $v = 4,\ b = 6,\ k = 2,\ r = 3$.



block 1 block 2 block 3 block 4 block 5 block 6

Balanced Incomplete block design (BIBD):- An arrangement of v treatments in b blocks of k plots each ($k < v$) is known as BIBD if

(1) each treatment occurs once and only once in r blocks,

(2) each pair of treatments occurs together in λ blocks i.e. λ is the number of blocks in which any pair of treatments occurs together.

In the above example $\lambda = 1$.

v, b, k, r and λ is known as the parameters of BIBD.

Incidence Matrix:- Associated with any design D, the incidence matrix is

$$N = (n_{ij})_{v \times b}, \quad \begin{matrix} i = 1, 2, \ldots, v. \\ j = 1, 2, \ldots, b. \end{matrix}$$

where n_{ij} denotes the no. of times the ith treatment occurs in the jth block.

i.e. $n_{ij} = \begin{cases} 1 & \text{if } i\text{th treatment occurs in } j\text{th block} \\ 0 & \text{otherwise} \end{cases}$

So, incidence matrix is a binary matrix.

12.04.

So, in the above example the incidence matrix is

$$N = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccccc} I & II & III & IV & V & VI \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

$v \times b$

Observations:-

1. $\sum_{j=1}^{b} n_{ij} = r \quad (i = 1, 2, \ldots, v)$

$= \sum_{j=1}^{b} n_{ij}^2$

2. $\sum_{i=1}^{v} n_{ij} = \sum_{i=1}^{v} n_{ij}^2 = k \quad (j = 1, 2, \ldots, b)$

$a_{11}\ a_{12}$
a_{21}

3. $\sum_{j=1}^{b} n_{ij} \cdot n_{lj} = \lambda \quad (i \neq l = 1, 2, \ldots, v)$ (in the above example $\lambda = 1$)

4. $(N \times N')_{v \times v} = \begin{pmatrix} r & \lambda & \lambda & \ldots & \lambda \\ \lambda & r & \lambda & \ldots & \lambda \\ \lambda & \lambda & r & \ldots & \lambda \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda & \lambda & \lambda & \ldots & r \end{pmatrix} = \lambda E_{vv} + (r - \lambda) I_v$,

where $E_{vv} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \ldots & 1 \end{pmatrix}$

Theorem:- The necessary condition for the existence of a BIBD are

13.04.2011

(i) $vr = bk$

(ii) $\lambda(v-1) = r(k-1)$

(iii) $b \geq v$ (Fischer's inequality)

4X 33
12XII

Proof :- (i) $N E_{b1} = r E_{v1}$

$\therefore E_{1v}(N E_{b1}) = r E_{1v} E_{v1} = rv \longrightarrow (1)$

Again, $E_{1v} N = k E_{1b}$

$\therefore (E_{1v} N) E_{b1} = k E_{1b} E_{b1} = kb \longrightarrow (2)$

From (1) & (2), we get

$rv = kb.$

(ii) $(NN') E_{v1} = \lambda E_{vv} E_{v1} + (r-\lambda) I_v E_{v1}$

$= \lambda v E_{v1} + (r-\lambda) E_{v1}$

$= (\lambda v + r - \lambda) E_{v1}$

$= [r + (v-1)\lambda] E_{v1}.$

Again, $N(N' E_{v1}) = N(k E_{b1}) = k N E_{b1} = k r E_{v1}$

$\therefore [r + \lambda(v-1)] E_{v1} = kr E_{v1}$

$\therefore r + \lambda(v-1) = kr$

$\therefore \lambda(v-1) = r(k-1).$

(iii) $|NN'| = \begin{vmatrix} r & \lambda & \lambda & \cdots & \lambda \\ \lambda & r & \lambda & \cdots & \lambda \\ \vdots & \vdots & & & \vdots \\ \lambda & \lambda & \cdots & \cdots & r \end{vmatrix}$

$= \begin{vmatrix} r+(v-1)\lambda & \lambda & \lambda & \cdots & \lambda \\ r+(v-1)\lambda & r & \lambda & \cdots & \lambda \\ \vdots & & & & \\ r+(v-1)\lambda & \cdots & \cdots & \lambda & r \end{vmatrix}$

$$= [r + (v-1)\lambda] \begin{vmatrix} 1 & \lambda & \lambda & \cdots & \lambda \\ 1 & r & \lambda & \cdots & \lambda \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda & \lambda & \cdots & r \end{vmatrix}$$

$$= [r + (v-1)\lambda] \begin{vmatrix} 1 & \lambda & \lambda & \cdots & \lambda \\ 0 & r-\lambda & 0 & \cdots & 0 \\ 0 & 0 & r-\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & r-\lambda \end{vmatrix}$$

$$= [r + (v-1)\lambda] \begin{vmatrix} r-\lambda & 0 & \cdots & 0 \\ 0 & r-\lambda & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & r-\lambda \end{vmatrix}$$

$$= [r + (v-1)\lambda] [r-\lambda]^{v-1}$$

$$= rk (r-\lambda)^{v-1}$$

Since, $v > k$

$\therefore v-1 > k-1$

$$\frac{v-1}{k-1} > 1$$

but $\frac{v-1}{k-1} = \frac{r}{\lambda}$

$\therefore r > \lambda$.

$\therefore |NN'| \neq 0$, non-singular.

$\therefore$ Rank $(NN') = v$.

Rank $(NN') \leq$ Rank (N)

$\therefore$ Rank $(N) \geq v$.

$\therefore b \geq v$. (Fischer's inequality)

Hence, the proof.

Corollary :- 1. $k \leq r$.

2. $b \geq v + r - k$.

proof :-

$v > k, \quad r \geq k$

$\therefore (v-k)(r-k) \geq 0$

$\Rightarrow \left(\frac{v}{k} - 1\right)(r-k) \geq 0$

$\Rightarrow \frac{vr}{k} - v \geq r - k$

$\Rightarrow b = \frac{vr}{k} \geq v + r - k.$

[Symmetric BIBD :- A BIBD is said to be symmetric if $b = v$ and $r = k$ i.e. the incidence matrix N is a square matrix.

Also, $|NN'| = r^2 (r-\lambda)^{v-1}$

Now, $|NN'| = |N||N'| = |N|^2$

$|N| = r (r-\lambda)^{\frac{v-1}{2}}$

Because $|N|$ is an integer when v is even $(r-\lambda)$ should be a perfect square.

28.04.2011

Theorem :- In a symmetric BIBD the number of treatment common between any two blocks is λ.

Proof :- We have,

$$NN' = (r-\lambda) I_v + \lambda E_{v\times v}$$

Now, cofactor of diagonal element of NN' is

$$\begin{vmatrix} r & \lambda & \lambda & \dots & \lambda \\ \lambda & r & \lambda & \dots & \lambda \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda & \lambda & \lambda & \dots & r \end{vmatrix}_{(v-1)\times(v-1)} = (r-\lambda)^{v-2} (r + \overline{v-2}\,\lambda)$$

$$= (r-\lambda)^{v-2} (rk - \lambda)$$

[$\because r + \overline{v-2}\,\lambda = r + \lambda(v-1) - \lambda = r + r(k-1) - \lambda = rk - \lambda$]

$$= r^2 (r-\lambda)^{v-2} - \lambda (r-\lambda)^{v-2} = A \text{ (say)}.$$ [$\because r = k$ for sym. BIBD]

The cofactor of off diagonal element NN' is

$$-\begin{vmatrix} \lambda & \lambda & \dots & \lambda \\ \lambda & r & \dots & \lambda \\ \lambda & \lambda & \dots & \lambda \\ \vdots & \vdots & & \\ \lambda & \lambda & \dots & r \end{vmatrix}_{(v-1)\times(v-1)} = -\begin{vmatrix} \lambda & 0 & \dots & 0 \\ \lambda & (r-\lambda) & \dots & 0 \\ \vdots & & & \\ \lambda & 0 & \dots & (r-\lambda) \end{vmatrix}_{(v-1)\times(v-1)}$$

$$= -\lambda (r-\lambda)^{v-2} = B \text{ (say)}.$$

Also, for a symmetric BIBD

$$|NN'| = r^2 (r-\lambda)^{v-1} = c \text{ (say)}$$

$$\therefore (NN')^{-1} = \begin{pmatrix} \frac{A}{C} & \frac{B}{C} & \cdots & \frac{B}{C} \\ \frac{B}{C} & \frac{A}{C} & \cdots & \frac{B}{C} \\ \vdots & \vdots & & \vdots \\ \frac{B}{C} & \frac{B}{C} & \cdots & \frac{A}{C} \end{pmatrix}$$

$$= \frac{1}{r-\lambda} \begin{pmatrix} 1-\frac{\lambda}{r^2} & -\frac{\lambda}{r^2} & \cdots & -\frac{\lambda}{r^2} \\ -\frac{\lambda}{r^2} & 1-\frac{\lambda}{r^2} & \cdots & -\frac{\lambda}{r^2} \\ \vdots & \vdots & & \\ -\frac{\lambda}{r^2} & -\frac{\lambda}{r^2} & \cdots & 1-\frac{\lambda}{r^2} \end{pmatrix}$$

$$= \frac{1}{(r-\lambda)} \left[I_v - \frac{\lambda}{r^2} E_{vv} \right]$$

Since, $\left[\lambda E_{vv} + (r-\lambda) I_v \right] \frac{1}{(r-\lambda)} \left[I_v - \frac{\lambda}{r^2} E_{vv} \right]$

$$= \frac{\lambda}{r-\lambda} E_{vv} - \frac{\lambda^2 v}{(r-\lambda) r^2} E_{vv} + I_v - \frac{\lambda}{r^2} E_{vv}$$

$$= I_v + \frac{\lambda}{(r-\lambda) r^2} \left[r^2 - \lambda v - (r-\lambda) \right] E_{vv}$$

$$= I_v + \frac{\lambda}{(r-\lambda)r^2}\left[r^2 - \lambda(v-1) - r\right]E_{vv}$$

$$= I_v + \frac{\lambda}{(r-\lambda)r^2}\left[r^2 - r(k-1) - r\right]E_{vv}$$

$$= I_v \qquad (\because r = k)$$

$$\therefore (N')^{-1} N^{-1} = \frac{1}{r-\lambda}\left[I_v - \frac{\lambda}{r^2}E_{vv}\right]$$

$$\therefore N^{-1} = \frac{1}{r-\lambda}\left[N' - \frac{\lambda}{r^2}N'E_{vv}\right], \text{ premultiplying by } N',$$

$$= \frac{1}{r-\lambda}\left[N' - \frac{\lambda}{r}E_{vv}\right] \qquad \left[\because N'E_{vv} = rE_{vv}\right]$$

$$I_v = \frac{1}{r-\lambda}\left[N'N - \frac{\lambda}{r}E_{v,v}N\right], \text{ post multiplying by } N.$$

$$= \frac{1}{r-\lambda}\left[N'N - \lambda E_{vv}\right] \qquad \left[E_{vv}N = rE_{vv}\right]$$

$$\therefore N'N = (r-\lambda)I_v + \lambda E_{vv}$$

$$= NN'.$$ ✓

Thus, the inner product of any two rows of N is equal to the inner product of any two columns of N i.e. λ.

Hence, the theorem]

Resolvable Design:- A BIBD with parameters v, b, k and λ is said to be resolvable if the b blocks can be ~~derived~~ devided into r groups or sets of $\frac{b}{r}$ blocks each. $\frac{b}{r}$ being an integer such that $\frac{b}{r}$ blocks forming any of this set gives a complete replications of all the v treatments.

e.g.:- Consider BIBD with $v, r, b = 6, r = 3, k = 2, \lambda = 1$.

$b = 6$ blocks are devided into $r = 3$ sets each of $\frac{b}{r} = 2$ blocks.

Now, the design is

Blocks	treatments	Sets / Groups
I	1 2	1st
II	3 4	
III	1 3	2nd
IV	2 4	
V	1 4	3rd
VI	2 3	

Theorem:- For a resolvable BIBD with parameter v, b, r, k, λ, $b \geq v + r - 1$ or equivalently, $k \leq r - \lambda$.

Proof:- Since, the design is resolvable, then

$\frac{b}{r} = n$ (an integer)

$\Rightarrow b = rn.$

$\therefore vr = bk = rnk \Rightarrow v = nk.$

Again, $r(k-1) = \lambda(v-1) = \lambda(nk-1)$

$\therefore r = \frac{\lambda(nk-1)}{k-1} = \lambda n + \frac{\lambda(n-1)}{(k-1)}$

$\therefore r - \lambda n = \frac{\lambda(n-1)}{(k-1)}.$

$\therefore \frac{\lambda(n-1)}{(k-1)}$ is an integer.

Now, if possible let $b < v + r - 1.$

i.e. $b - r < v - 1$

$\Rightarrow r(n-1) < v - 1$ [$\because b = nr$]

$\Rightarrow r(n-1) < \frac{r(k-1)}{\lambda}$

$\Rightarrow \frac{\lambda(n-1)}{k-1} < 1$

but $n > 1, k > 1,$

a contradiction.

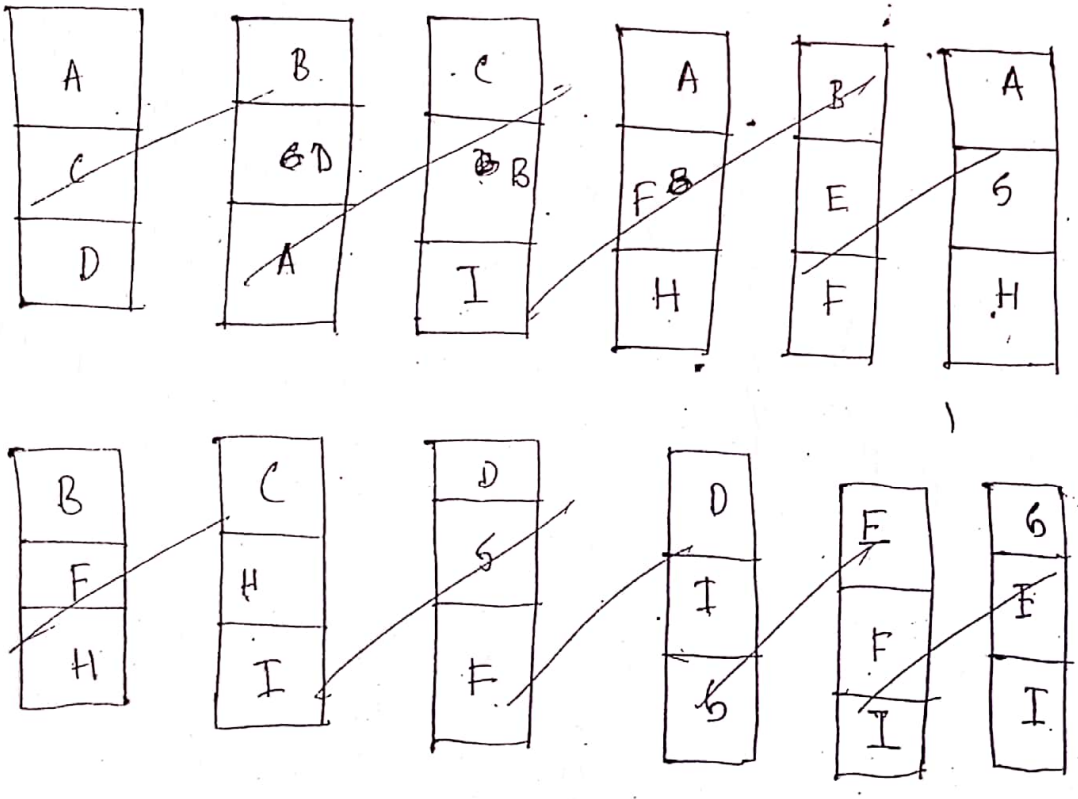
ne resolvable design :- A resolvable design said to be affine resolvable design if $b = v + r - 1$, or equivalently, any two blocks from different sets have $\frac{k^2}{v}$ treatments common where $\frac{k^2}{v}$ is an integer.

For previous example, $v = 4$, $b = 6$, $r = 3$, $k = 2$, $\lambda = 1$, $b = v + r - 1$.

Also, $\frac{k^2}{v} = 1$ and any blocks from different sets have only one treatments ($\lambda = 1$) common.

e.g.:- Construct a ~~A bd balanced~~ BIBD with parameters $v = 9$, $b = 12$, $k = 3$, $r = 4$, $\lambda = 1$ with incidence matrix.

A B C D E F G H I



2 3

5 6

7 8 9

1 4 7

2 5 8

3 6 9

1 6 8

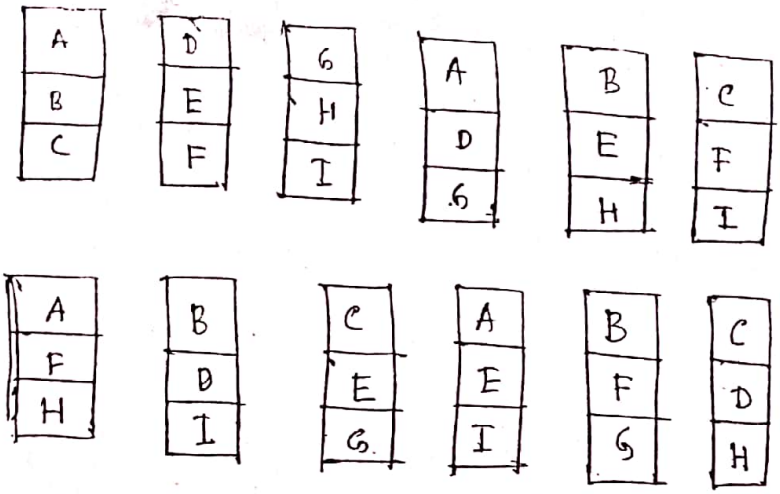
2 4 9

3 5 7

1 5 9

2 6 7

3 4 8



Now, we shall want to construct the incidence matrix corresponding to the given BIBD:-

$$\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0
\end{bmatrix}$$

Ex :-

(1, 3, 4, 5, 9), (2, 4, 5, 6, 10), (3, 5, 6, 7, 11), (4, 6, 7, 8, 1), (5, 7, 8, 9, 2), (6, 8, 9, 10, 3), (7, 9, 10, 11, 4), (8, 10, 11, 1, 5), (9, 11, 1, 2, 6), (10, 1, 2, 3, 7), (11, 2, 3, 4, 8).

Find the parameters. v, b, k, r, λ.

$v = 11$, $b = 11$, $k = 5$, $r = 5$, $\lambda = 2$.

Construction of BIBD :- There is no single general method of construction of all BIBD. There are specific methods for specific series of design. Solutions of many design are not yet known.

BIBD with parameters v, $b = \frac{v(v-1)}{2} = \binom{v}{2}$, $k = 2$, $r = (v-1)$, $\lambda = 1$, $v > 2$. So, all possible combinations of v treatments taken 2 at a time are used as blocks. By repeating p times the b blocks of a BIBD with parameters v, b, r, k and λ. Another BIBD with parameters v, pb, pr, k and $p\lambda$. is obtained. Such designs are called repeated design. Similarly we can construct design with parametres v, k, $b = {}^{v}C_{k}$, $r = {}^{v-1}C_{k-1}$, $\lambda = {}^{v-2}C_{k-2}$. it is obtained by taking all possible combinations of v treatments taken k at a time at each

[inequality relation]

blocks. Another BIBD with parameters v, k, $b \cdot {}^{v}C_{k}$, $b \cdot {}^{v-2}C_{k-2}$ and $b \cdot {}^{v-2}C_{k-2}$.

25.04.2011

Some exercise:-

1. If N is the incidence matrix of a symmetric BIBD then prove that,

$$(NN')(N'N) = (r-\lambda) N'N + k^2 \lambda E_{vv}.$$

2. Show that the BIBD with parameter (i) $v = b = 22$, $r = k = 7$, $\lambda = 2$ and (ii) $v = b = 34$, $r = k = 12$, $\lambda = 4$ does not exist.

3. Show that in a symmetric BIBD any 2 blocks have the same no. of treatments in common.

4. Show that if from a symmetric BIBD delete one block and all treatments belonging to the block, then the reminder is again BIBD. Find it's parameter.

Symmetric BIBD

5. Show that, if x is a no. of treatments common between any two blocks of a BIBD with parametres v, b, r, k, λ, then

$$\frac{2k\lambda}{r} + \frac{r(r-\lambda-k)}{r} \geq x \geq \lambda - (r-k-\lambda).$$

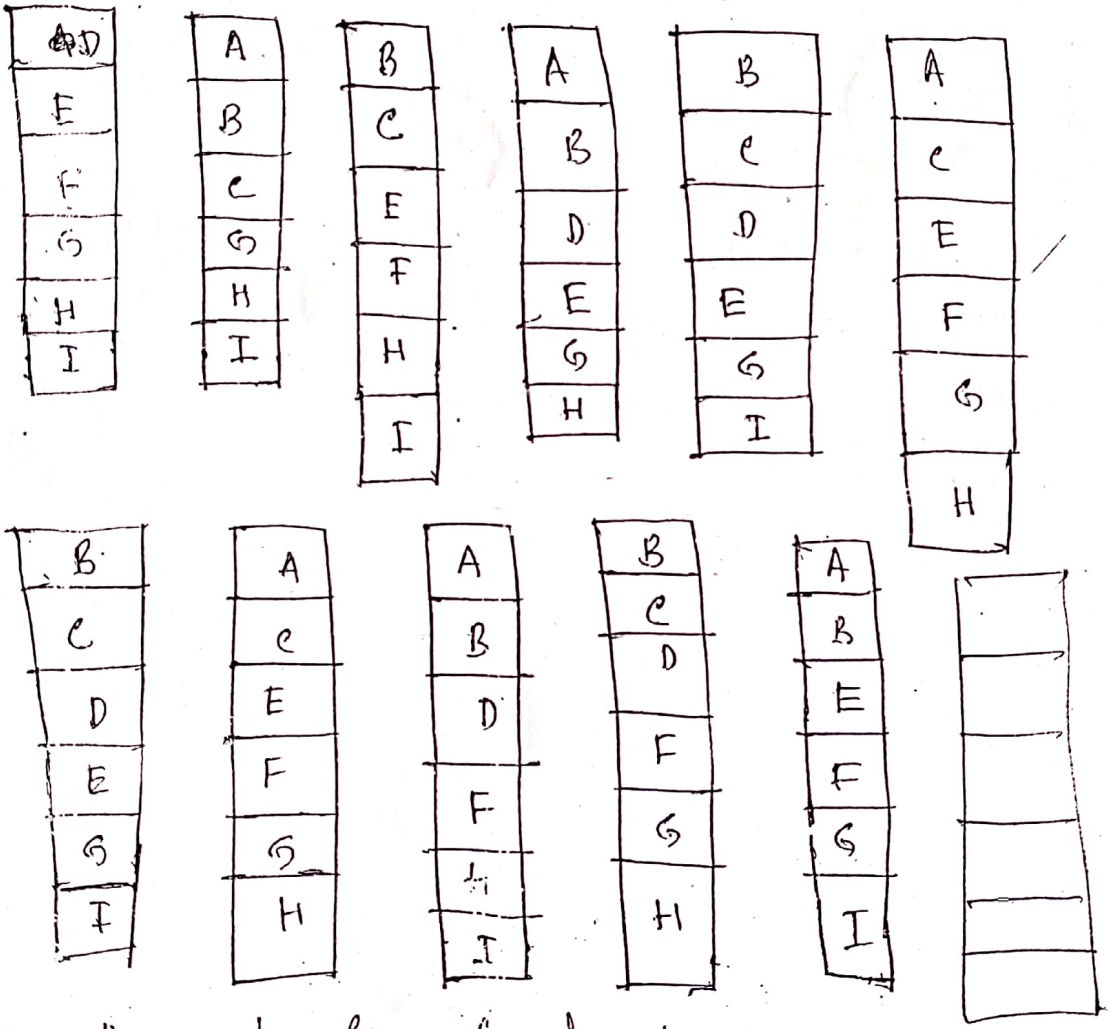
6. If N is a incidence matrix of a BIBD with parameters v, b, r, k, λ, then show that,

$$(NN')(N'N) = (r-\lambda) N'N + rk\lambda E_{vv}.$$

Complementary Design (C.D.)

Given a BIBD, D with parameters v, b, r, k, λ, another BIBD called the C.D. of D denoted by $\bar{D}$ is obtained from D by taking in the jth block of $\bar{D}$ all those treatments which do not occurs in jth block of D, for $j = 1, 2, \ldots, b$.

e.g.:- Construct a complementary design $\bar{D}$ with parameters $v = 9$, $b = 12$, $k = 6$, $r = 8$, $\lambda = 5$, for its BIBD (D).



Hence, the parameters of complementary design is $v_1 = 9$, $b_1 = 12$, $k_1 = 6$, $r_1 = 4$, $\lambda_1 = 3$.

Obviously, if $v_1, b_1, r_1, k_1, \lambda_1$ are the parameters of $\bar{D}$, then $v_1 = v$, $b_1 = b$,

$$k_1 = v - k, \quad r_1 = \frac{b_1 k_1}{v_1} = \frac{b(v-k)}{v} = b - \frac{bk}{v} = b - r$$

$$\lambda_1 = \frac{r_1(k_1-1)}{v_1-1} = \frac{r_1 k_1 - r_1}{v_1-1} = \frac{b(v-k)}{v} \cdot \frac{(v-k-1)}{v-1}$$

λ, r, b

$$= \frac{(bv-bk)(v-k-1)}{v^2-v} = \frac{bv^2 - bvk - bv - bvk + bk^2 + bk}{v^2-v}$$

$$= \frac{(b-r)(v-k-1)}{v-1} = \frac{b(v^2-v) - 2bvk + bk(k+1)}{v^2-v}$$

$$= \frac{bv - bk - b - vr + kr + r}{v-1} = b - \frac{2bk}{v-1} + \frac{bk(k+1)}{v(v-1)} = \frac{r(k-1)}{v-1}$$

~~$= b(v-k-1) + r(k$~~

$$= b - \left\{\frac{2bvk + bk^2 - bk}{v(v-1)}\right\}$$

$$= \frac{r(k-1) + 2r - vr + bv - bk - b}{v-1}$$

$$= \lambda + \frac{r(2-v)}{v-1} + \frac{b(v-k-1)}{v-1}$$

$$= \lambda + \frac{r\left(2 - \frac{bk}{r}\right)}{\frac{bk}{r} - 1} + \frac{b\left(\frac{bk}{r} - k - 1\right)}{\frac{bk}{r} - 1}$$

$$= \lambda + \frac{r^2\left(2 - \frac{bk}{r}\right)}{bk - r} +$$

$$= (k - 2r + \lambda)$$

~~Obiviously~~,

Let, $\bar{N}$ is the incidence matrix of D.

Then,

$\bar{N} = JJ' - N$.

$\bar{N}J = (b-r)J$

$J'\bar{N} = (v-k)J'$

$\therefore \bar{N}\bar{N}' = (r-\lambda)I + (b-2r+\lambda)JJ'$.

$\sigma = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{v\times 1}$

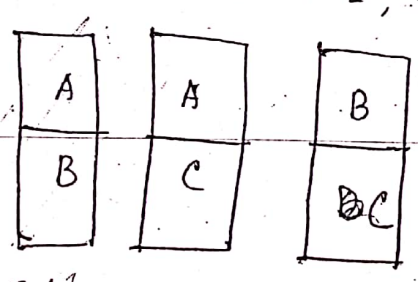
[incidence matrix ও যেখানে 1 ছিল সেখানে 0 হয়ে যাবে, যেখানে 0 ছিল সেখানে 1 হবে]

Residual design:- Consider a symmetric BIBD, D with parameters v, k, λ. Choose any block of D and delete from D the chosen block and all treatment contained in the chosen block. The design, D_2 containing the remaining blocks is known as residual design. This process of getting a BIBD from a symmetric BIBD is called "block section". Then, D_2 is a BIBD with parameters $v_2 = (v-k)$, $b_2 = (v-1)$, $r_2 = k$, $k_2 = (k-\lambda)$, $\lambda_2 = \lambda$.

~~Derived~~

Eg:- Give an example of symmetric design and delete the 1st block to get a residual design.

Let, $b = v = 3$, $k = r = 2$ and $\lambda = 1$, then we have a symmetric BIBD



$A = v^{(1)}$

$B = v^{(2)}$

$C = v^{(3)}$

Derived design:- Starting from a symmetric BIBD, D with parameters v, k, λ, we can delete a block of D and written only those treatment in other blocks of D, which appears in the deleted block. Let, the resultant design is called derived design, D_3. The no. of treatments in D_3 is $v_3 = k$, $b_3 = b - 1$ ($b = v$). Also, since only 2 blocks of D have λ treatment in common, the block size in D_3 is $k_3 = \lambda$. Since, each of the written treatments appears once in the deleted block, the replication of each treatments is $r_3 = k - 1$. Similarly, each pair of treatments appears together in D_3 is $\lambda_3 = \lambda - 1$.

This process is known as "Block-intersection".

Theorem 1:- If a BIBD with parameters v, b, r, k, λ is resolvable, then $b \geq v + r - 1$, and $b = v + r - 1$ iff $\frac{k^2}{v}$ be the no. of treatments common between any two blocks of the different sets.

e.g.:- Consider the BIBD with parameters $v = 10$, $b = 18$, $r = 9$, $k = 5$, $\lambda = 4$. Here, $\frac{b}{r} = 2$ and $b = v + r - 1$, but $\frac{k^2}{v} = \frac{25}{10} \notin \mathbb{Z}$.

Hence, if the design is resolvable this must be affine resolvable and $\frac{k^2}{v} \in \mathbb{Z}$.

Therefore, the design is not resolvable.

Theorem 2:- The parameters of an affine resolvable BIBD can be expressed interms of two positive integers.

Proof:- Let, v, b, r, k, λ be the parameters of an affine resolvable BIBD.

Let, $m = \frac{k^2}{v}$, an integer.

Hence, $k = mn$, where $n = \frac{b}{r}$.

Therefore, $v = mn^2$, $b = nr$, $k = mn$

Now, $nr = mn^2 + r - 1$ [নরীকরণ detail এইগুলো লিখতে হবে।]

or, $r = \frac{mn^2 - 1}{n - 1} = mn + m + \frac{m-1}{n-1}$

Also, $\lambda = \frac{r(mn-1)}{mn^2-1} = m + \frac{m-1}{n-1}$.

Again put, $\lambda - m = \frac{m-1}{n-1} = t$,

then $t \geq 0$, an integer.

and $m = (n-1)t + 1$,

Thus, $v = \{(n-1)t+1\}n^2$

$b = n(n^2t + n + 1)$

$r = n^2t + n + 1$

$k = n\{(n-1)t + 1\}$

$\lambda = nt + 1$.

Exercise:

1. Prove that, for a BIBD, $b \geq v + r - k$. Is this inequlity & equivalent to Fisher's inequality?

2. Let, N be the incidence matrix of a BIBD, show that,

(i) $\det(N'N) = 0$ if $b > v$ and

(ii) the eigen values of NN' are rk and $r-\lambda$ with multiplicity unity and $v-1$ respectively.

Further construction of BIBD:- Construction of BIBD with parameters

$v = s^2$, $b = s^2 + s$, $k = s$, $r = s + 1$, $\lambda = 1$,

where s is either a prime or a prime power.

$s = p^n$ where p is prime, $n \in N$

The above series of design can be obtained by first forming a finite 2 dimensional euclidian geometry by using the elements ~~GF(2) GF(p^n)~~ $GF(s)$ and then treating the points as treatments. All possible lines as blocks and points on a line as the contains of the block corresponding to the line.

e.g. When $s = 4$, then $v = 16$, $b = 20$, $k = 4$, $r = 5$, $\lambda = 1$.

The elements in the $GF(4)$, with 4 elements are $0, 1, \alpha, \alpha^2$ with $\alpha^2 + \alpha + 1 = 0$. There are 16 points in the two dimensional euclidian geometry which can be written using the elements. The following 16 possible pairs (x, y) form of these elements

p prime $\{0, 1, 2, \ldots, p-1\}$

$a + b \equiv c \pmod p$

$a \cdot b \equiv d \pmod p$

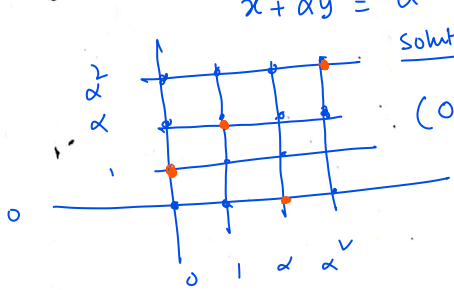
$\alpha^2 \cdot \alpha = 1$

~~$(x, y) = (0, 1, \alpha, \alpha^2)$~~

$x = 0, 1, \alpha, \alpha^2$

$y = 0, 1, \alpha, \alpha^2$

$x + \alpha y = \alpha$ solutions



$(0, 1)$, $(1, \alpha)$

(α^2, α^2)

$(\alpha, 0)$

give the 16 pts as follows :-

Point No:-

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
pt: 00	01	0α	$0\alpha^2$	10	11	1α	$1\alpha^2$	$\alpha 0$	$\alpha 1$	$\alpha\alpha$	$\alpha\alpha^2$	$\alpha^2 0$	$\alpha^2 1$	$\alpha^2 \alpha$	$\alpha^2\alpha^2$

The equns of the possible lines are:

(1) $x = i$, $(i = 0, 1, \alpha, \alpha^2)$

(2) $y = i$, $(i = 0, 1, \alpha, \alpha^2)$

(3) $x + y = i$, $(i = 0, 1, \alpha, \alpha^2)$

(4) $x + \alpha y = i$, $(i = 0, 1, \alpha, \alpha^2)$

(5) $x + \alpha^2 y = i$, $(i = 0, 1, \alpha, \alpha^2)$

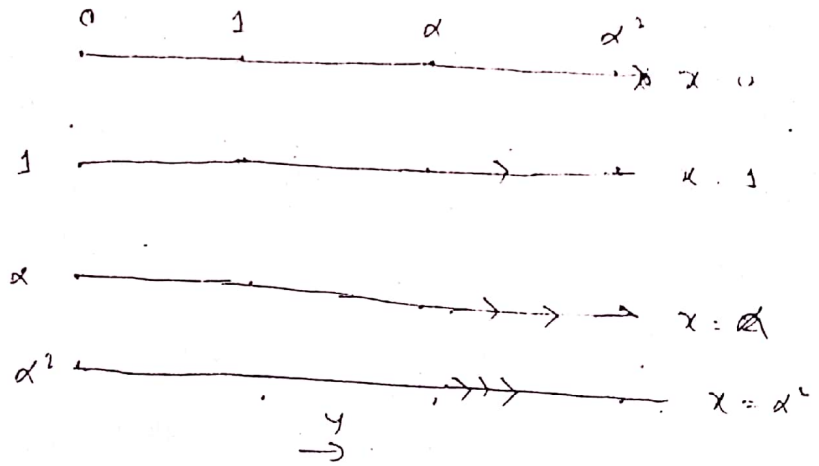
$x = 1$: $(0, 1, 1, 1\alpha, 1\alpha^2)$

$\alpha x + y = 0$ } same line
$x + \alpha^2 y = 0$

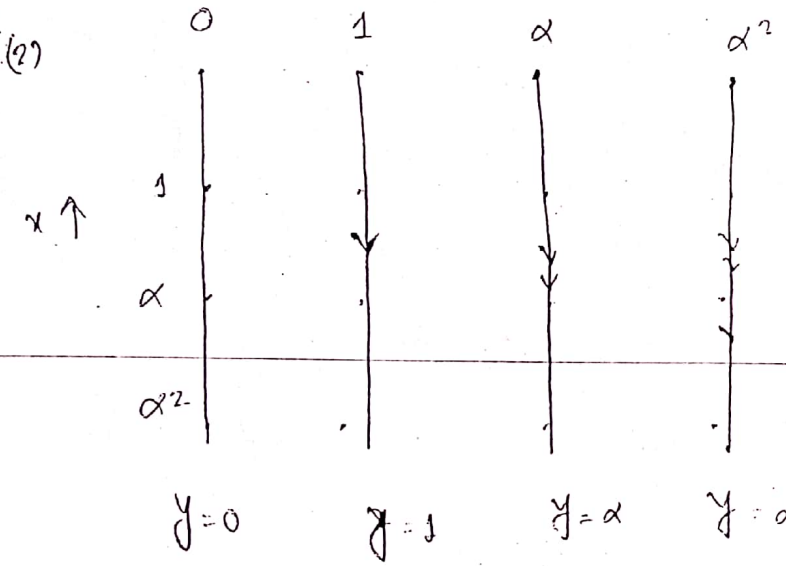
$\alpha x + y = 1$ } same line
$x + \alpha^2 y = \alpha^2$

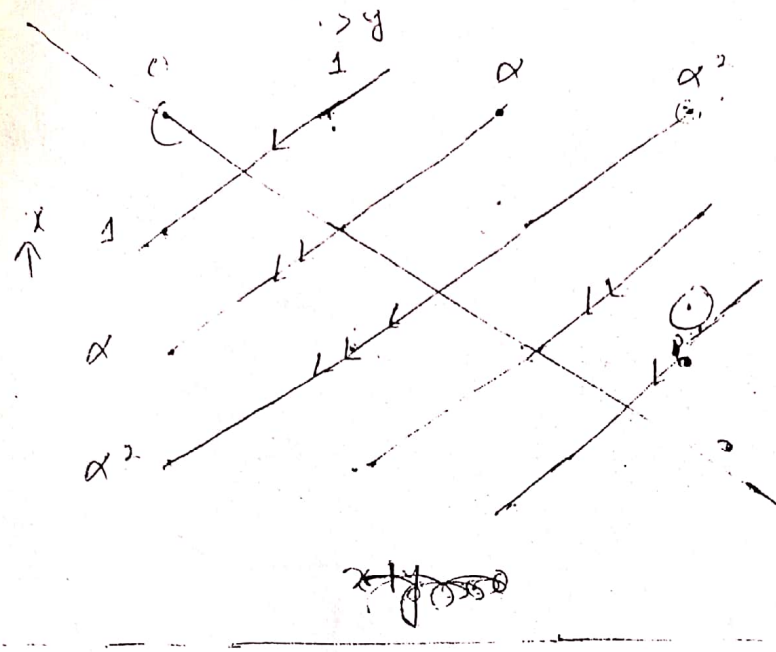
$x + \alpha y = 1 + \alpha$ } same line
$x + \alpha y = \alpha^2$

(1)



(2)





$x+\alpha^2 y = 0$ α^2 $(0,0), (\alpha^2,1), (1,\alpha), (\alpha,\alpha^2)$

$(1,0), (\alpha,1), (\alpha^2,\alpha^2), (0,\alpha)$

$(\alpha,0), (\alpha^2,\alpha), (0,\alpha^2), (1,1)$

$(\alpha^2,0), (0,1), (1,\alpha^2), (\alpha,\alpha)$

$x+\alpha y = \alpha^2$

$(0,\alpha), (\alpha^2,0), (1,1), (\alpha,\alpha^2)$

$1+\alpha$ α $x+\alpha y = \alpha$

$1,1$ $x+\alpha y = 0$ $x+\alpha y = 1$ $1+\alpha+\alpha^2 = 0$

$\Rightarrow \alpha+\alpha^2 = -1$

$x=0, y=0$.

$1+\alpha, \alpha$ α, α $(0,0), (\alpha^2,\alpha), (\alpha,1)$,

$(1,0)$ $(1,\alpha^2)$.

$1, 1+\alpha$. $(0,\alpha)$.

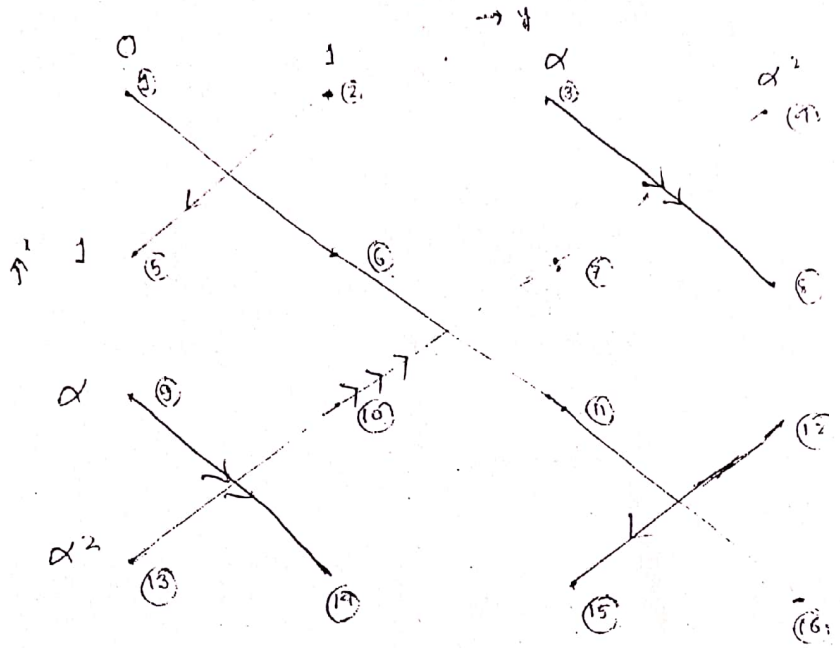
$\alpha^2+1, 1$.

$(1,0)$

$(0,\alpha^2)$

(α,α) $(\alpha^2,\alpha+1)$ $x+\alpha(1+\alpha)$

$= 0+\alpha+\alpha^2 = 1$



Thus, there are 5 set of line with 4 parallel lines in each set. There is no common point in any two lines belonging to a set. In general, there are $\infty(s+1)$ such set of lines each set containing s parallel line. So, the total no. of line is $s \times (s+1)$. The no. of pts in a line, say $x + py = i$ is s. where $p \neq i$ are two elements in the GF(s) i.e. so because for each value of x in the field only one value of y can be obtained from the equn. Given a point, there is one line in each set which passes through the point. As there are $(s+1)$ set, $(s+1)$ line passes through the point. Hence, $r = s+1$. Only, one line can pass through the point i.e. $\lambda = 1$.

$x = 0$

Block No.	Contents	Eqn	solution $00, 01, 0\alpha, 0\alpha^2$
			1 2 3 4
1	1 2 3 4	$x = 0$	
2	5 6 7 8	$x = 1$	
3	9 10 11 12	$x = \alpha$	
4	13 14 15 16	$x = \alpha^2$	
5	1 5 9 13	$y = 0$	
6	2 6 10 14		
7	3 7 11 15		
8	4 8 12 16		
9	1 6 11 16		
10	2 5 12 15		
11	3 8 9 14		
~~12~~	~~13 10 7 4~~		
12	1 7 10 13		
13	1 8 10 15		
14	4 5 11 14		
15	2 7 9 16		
16	3 6 12 13		
17	1 7 12 14		

18	3	5	10	16
19	4	6	9	15
20	2	8	11	13

When $s = 6$, no GF exists.

Hence, the design for $s = 6$, $v = 36$, $b = 42$, $r = 7$, $k = 6$, $\lambda = 1$ is not obtainable by this method.

Actually, it can be shown that, this design is impossible. By taking $(s+1)$ more treatments and by putting ith of this treatments (when arrange in some order) in each of the s blocks in the ith set of block $(i = 1, \ldots, s+1)$. and all these treatments in a new block, we get a symmetric BIBD with (s^2+s+1) treatments distributed over (s^2+s+1) blocks each of size $r = (s+1)$ ~~r = s+1~~, $\lambda = 1$.

Treatments
$s^2+(s+1)$

blocks
s^2+s+1

An alternative way obtaining a solution of the design of the series $v = s^2$, $b = s^2+s$, $r = s+1$, $k = s$ & $\lambda = 1$ based on a complete set of MOLS :- It is known that the no. of latin square in a set of MOLS of order

s is almost $(s-1)$. When, there are $(s-1)$ square in a set of MOLS of order s, we say that we have a complete set of MOLS. When, s is prime or prime power, there exists a complete set of MOLS.

Let, the s^2 treatments arrange in a square array say, L_0 in any order and suppose, there exists a complete set of MOLS of order s, denoted by say, $L_0, L_1, L_2, \ldots, L_{s-1}$. Obtain s incomplete blocks, each of size s by treating the rows of L_0 as block. This form the first replication. Another s blocks are obtained by treating the columns of L_0 as blocks.

Superimpose one of the LS, say, L_i, $i=1(1)s$ on S_0 and formed a block by including all those treatments which fall under a given letter of L_i. Repeat this procedure, with all the LS in the set. This procedure generates a totality of $b = 2s + s(s-1)$

$= 2s + s^2 - s = s^2 + s$

$= s(s+1)$

blocks in all.

Eg.:- For $S = 2^2$

$$L_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 0 & 1 & \alpha & \alpha^2 \\ 1 & 0 & \alpha^2 & \alpha \\ \alpha & \alpha^2 & 0 & 1 \\ \alpha^2 & \alpha & 1 & 0 \end{pmatrix}$$

$$L_2 = \begin{pmatrix} 0 & 1 & \alpha & \alpha^2 \\ \alpha & \alpha^2 & 0 & 1 \\ \alpha^2 & \alpha & 1 & 0 \\ 1 & 0 & \alpha^2 & \alpha \end{pmatrix}$$

$$L_3 = \begin{pmatrix} 0 & 1 & \alpha & \alpha^2 \\ \alpha^2 & \alpha & 1 & 0 \\ 1 & 0 & \alpha^2 & \alpha \\ \alpha & \alpha^2 & 0 & 1 \end{pmatrix}$$

Theorem :- By taking the points of a projective geometry PG (n, s) as treatments and the m flats of it as blocks we get a BIBD with the following parameters

$$v = \frac{s^{n+1} - 1}{s - 1}$$

$$b = \frac{(s^{n+1} - 1)(s^n - 1) \cdots (s^{n-m+1} - 1)}{(s^{m+1} - 1)(s^m - 1) \cdots (s - 1)}$$

$$r = \frac{(s^n - 1)(s^{n-1} - 1) \cdots (s^{n-m+1} - 1)}{(s^m - 1)(s^{m-1} - 1) \cdots (s - 1)}$$

$$k = \frac{(s^{m+1} - 1)}{(s - 1)}$$

$$\lambda = \frac{(s^{n-1} - 1)(s^{n-2} - 1) \cdots (s^{n-m+1} - 1)}{(s^{m-1} - 1)(s^{m-2} - 1) \cdots (s - 1)}$$

where $s = p^h$, p is prime.

Proof :- Consider a PG (n, s) to every point in PG (n, s) let there correspond a treatment. Next, let an m-flat in the geometry be identified with a block.

Then the no. of treatments & blocks are respectively given by $v = \phi(n, 0, s)$ =

$$= \frac{s^{n+1} - 1}{s - 1}$$

$b = \phi(n, m, s)$ = the no. of m flats in $PG(n, s)$

The no. of points in each m flat i.e. the no. of treatments contained in each block is

$$k = \phi(m, 0, s) = \frac{s^{m+1} - 1}{s - 1}$$

Each treatment is contained in r blocks, where r is the no. of m flat passing through a point.

Thus, $r = \phi(n-1, m-1, s)$

$$= \frac{(s^n - 1)(s^{n-1} - 1) \cdots (s^{n-m+1} - 1)}{(s^m - 1)(s^{m-1} - 1) \cdots (s - 1)}$$

Similarly, each pair of treatments is contained in λ blocks,

λ is the no. of m flats passing through a given pair of pts. and hence

$$\lambda = \phi(n-2, m-2, s)$$

$$= \frac{(s^{n-1} - 1)(s^{n-2} - 1) \cdots (s^{n-m+1} - 1)}{(s^{m-1} - 1)(s^{m-2} - 1) \cdots (s - 1)}$$

Some Series:

Series no. 1 :- $n = 2, \; m = 1$

$$v : \frac{s^3 - 1}{s - 1} = (s^2 + s + 1)$$

$$b : \frac{(s^3 - 1)(s^2 - 1)}{(s^2 - 1)(s - 1)} = (s^2 + s + 1)$$

$$= (s^2 + s + 1)$$

$$p = \frac{(s^2 - 1)(s - 1)}{(s - 1)} = (s + 1).$$

$$k = \frac{(s^2 - 1)}{(s - 1)} = (s + 1).$$

$$\lambda = \frac{(s - 1)(s^2 - 1)}{s} = \infty \cdot 1$$

$$s = b^2,$$

Series. no. 2 :- $n = 3, \; m = 2.$

$$v = \frac{s^4 - 1}{s - 1} = \frac{(s^2 + 1)(s + 1)(s - 1)}{(s - 1)} = (s^2 + 1)(s + 1)$$

$$b = \frac{(s^4 - 1)(s^2 - 1)(s^3 - 1)}{(s^3 - 1)(s^2 - 1)(s - 1)} = (s^3 + s^2 + s + 1)$$

$$r = \frac{(s^3-1)(s^2-1)(s-1)}{(s^2-1)(s-1)(s-1)} = \frac{(s-1)(s^2+s+1)}{(s-1)}$$

$$= (s^2+s+1).$$

$$k = \frac{(s^3-1)}{(s-1)} = (s^2+s+1)$$

$$\lambda = \frac{(s^2-1)(s-1)}{(s-1)} = (s+1)$$

H. T. Series Number 3:

$n = 3, \quad m = 1.$

a. $\frac{s^4-1}{s-1} = \frac{(s^2+1)(s^2-1)}{(s-1)} = \frac{(s^2+1)(s+1)(s-1)}{(s-1)}$ $\quad s^3 + s^2 + s + 1$

b. $\frac{(s^4-1)(s^3-1)}{(s^2-1)(s-1)} = \frac{(s^2+1)(s+1)(s-1)(s-1)(s^2+s+1)}{(s+1)(s-1)(s-1)}$

$= (s^2+1)(s^2+s+1) = s^4 + s^3 + s^2 + s^2 + s + 1$

$\therefore s^4 + s^3 + 2s^2 + s + 1.$

$r = \frac{(s^3-1)}{(s-1)} = (s^2+s+1)$

$k = \frac{s^2-1}{s-1} = (s+1).$

$\lambda = 1.$

Eg:- $|c|$, $s=2$ in the series 2.

$v = 15 = b$.

$r = 7 = k$.

$\lambda = 3$. $\therefore m = 2, n = 3$.

In PG (3,2) each point is represented by ~~four~~ or 1-tuple (x_0, x_1, x_2, x_3), where each x_i is 0 or 1.

The two flats in PG(3,2) have equations

$x_i = 0$, $i = 0, 1, 2, 3$.

$x_i + x_j = 0$, $i, j = 0, 1, 2, 3$, $i \neq j$.

$x_i + x_j + x_k = 0$, $i, j, k = 0, 1, 2, 3$, $i \neq j \neq k$

$x_0 + x_1 + x_2 + x_3 = 0$.

If we associate the treatment

$8x_0 + 4x_1 + 2x_2 + x_3$ to the points (x_0, x_1, x_2, x_3)

The treatments of the design are now numbered as $1, 2, \ldots, 15$. The blocks ~~are~~ of these 7 treatments are as follows:-

B1 (1, 2, 3, 4, 5, 6, 7)

B2 (1, 2, 3, 8, 9, 10, 11),

B3 (1, 5, 4, 8, 9, 12, 13),

B4 (2, 4, 6, 8, 10, 12, 14),

B5 (1, 2, 3, 12, 13, 14, 15).

B6 (1, 4, 5, 10, 11, 14, 15)

B7 (2, 4, 6, 9, 11, 13, 15)

B8 (1, 6, 7, 8, 9, 14, 15)

B9 (2, 5, 7, 8, 10, 13, 15)

B10 (3, 4, 7, 8, 11, 12, 15)

B11 (1, 6, 7, 10, 11, 12, 13)

B12 (2, 5, 7, 9, 11, 12, 14)

B13 (3, 4, 7, 9, 10, 13, 14)

B14 (3, 5, 6, 8, 11, 13, 14)

B15 (3, 5, 6, 9, 10, 12, 15)

Theorem :- By taking points euclidean geometry, EG (n, s) as treatments and m flats as blocks we get a BIBD with parameters

$v = s^n$

$$b = \frac{s^{n-m}(s^n-1)(s^{n-1}-1)\cdots(s^{n-m+1}-1)}{(s^m-1)(s^{m-1}-1)\cdots(s-1)}$$

$k = s^m$

$$r = \frac{(s^n-1)(s^{n-1}-1)\cdots(s^{n-m+1}-1)}{(s^m-1)(s^{m-1}-1)\cdots(s-1)}$$

$$a = \frac{(s^{n-1}-1)(s^{n-2}-1)\cdots(s^{n-m+1}-1)}{(s^{m-1}-1)(s^{m-2}-1)\cdots(s-1)}$$

Series I :- $n = 2,\ m = 1.$

Series II :- $n = 3,\ m = 2.$

for Series II :-

$b = s^3,$

$$b = \frac{s^{3-2}(s^3-1)(s^{3-1}-1)}{(s^1-1)(s^2-1)}$$

$$= \frac{s(s^3-1)(s^2-1)}{(s-1)(s^2-1)} = s(s^3-1)(s+1)$$

$$= (s^2+s)(s^3-1) = s(s^2+s+1)$$

$$= s^3+s^2+s.$$

$k = s^2$

$$r = \frac{(s^3-1)(s^2-1)}{(s^2-1)(s-1)} = (s+1)(s^2+s+1)$$

$$= (s^2+s+1).$$

$$a = \frac{(s^2-1)(s-1)}{(s-1)} = (s+1).$$

Ist Series :- $n = 2, m = 1$

$$v = s^2$$

$$b = \frac{s(s^2-1)}{(s-1)} = s(s+1)$$

$$k = s.$$

$$r = \frac{(s^2-1)}{(s-1)} = (s+1)$$

$$\lambda = \frac{(s-1)}{0} = 1.$$

e.g. $v = 8,\ b = 14,\ r = 7,\ k = 4,\ \lambda = 3.$

$\therefore n = 3, m = 2, s = 2.$

∴ Identifying the points of EG(3,2) as treatments & plans (? flat) as blocks.

To get the following sol:s of BIBD.

Here, the point (x_1, x_2, x_3) is identified with the treatment

$$1x_1 + 2x_2 + x_3$$

where $x_i = 0$ or 1.

(0, 2, 1, 3)

(4, 6, 5, 7)

(0, 4, 1, 5)

(2, 6, 3, 7)

(0, 4, 2, 6)

~~(0, 5, 2, 7)~~

(1, 5, 3, 7)

(0, 3, 4, 7)

(2, 4, 6, 5)

(0, 5, 2, 7)

(4, 4, 6, 3)

(0, 6, 4, 7)

(4, 2, 5, 3)

(0, 3, 5, 6)

(4, 2, 6 7)

Theorem :- Let, $v \equiv 1 \pmod 4$. Then a necessary condition for the existence of a symmetric BIBD design with parameters $v = b$, $r = k$, λ is that the diophantine equation

$$x^2 \equiv (r-\lambda) y^2 + \lambda z^2$$

has a solution in integers, not all zero.

Theorem:- Let $v = 3 \pmod 4$ then a necessary condition for the existence of a symmetric BIBD with parameters $v = b$, $r = k$, λ is that the diophantine equation

$$x^2 = (r-\lambda)y^2 - \lambda z^2$$

has a solution in integers, not all zero.

Corollary:- A necessary condition for the existence of a symmetric BIBD with parameters $v = b$, $r = k$, λ, where v is odd is that the diophantine equation

$$x^2 = (r-\lambda)y^2 + (-1)^{\frac{v-1}{2}} \lambda z^2$$

has a solution in integers, not all zero.

Theorem:- Let u be the square-free part of $(r-\lambda)$ and t be the square-free part of λ, then for the existence of a symmetric BIBD with parameters $v = b$, $r = k$, λ and v is odd, it is necessary that

(i) for all odd primes p dividing v but not t,

$$\left((-1)^{\frac{v-1}{2}} t \mid p\right) = 1$$

(ii) for all odd primes p dividing t but not v,

$$\left(v \mid p\right) = 1$$

(iii) for all odd primes dividing both t and v

$$\left(-(-1)^{\frac{v-1}{2}} \mid p\right) = 1.$$

where $v_0 p = v$, $t_0 p = t$

Recall that the Legendre symbol

$$(b/p) = \begin{cases} 1 & \text{if } b \text{ is a quadratic residue modulo } p \\ -1 & \text{if } b \text{ is non-residue modulo } p \end{cases}$$

$x^2 \equiv a \pmod p$

a is quadratic residue of mod p

e.g.- Following BIBD does not exists

1) $b = v = 29$, $r = k = 8$, $\lambda = 2$

2) $b = v = 67$, $r = k = 12$, $\lambda = 2$

(1) $r - \lambda = 6 = u$ (say), $\lambda = 2 = t$ (say)

u is the square free part of $r - \lambda$.

$t = 2$ is the " " " " λ.

Let, $p=3$. $3 \mid u$. but $3 \nmid t$.

Now, $\left((-1)^{\frac{v-1}{2}} t \mid p\right)$

$= (t \mid p)$

$= (2 \mid 3)$

$= (-1)^{\frac{3^2-1}{8}} = -1 \neq 1$.

(2) $(r-\lambda) = 10 = u$

$t = 2$

Here $p = 5$

$\left((-1)^{\frac{v-1}{2}} t \mid p\right)$

$= (-2 \mid 5) = (-1 \mid 5)(2 \mid 5) = (-1)^2 (1)^3 =$

$= -1 \neq 1$

Formula of Legendre's :-

Notes :- (1) $(-1 \mid p) = (-1)^{\frac{p-1}{2}}$

(2) $(2 \mid p) = (-1)^{\frac{p^2-1}{8}}$

(3) $(p \mid q)(q \mid p) = (-1)^{\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)}$

where p, q are distinct odd pair

(4) $b \equiv c \pmod p$

$\Rightarrow (b \mid p) = (c \mid p)$

(5) $b^{\frac{p-1}{2}} \equiv (b \mid p) \pmod p$

$(ab \mid p) =$

Corollary :- If $s \equiv 1$ or $2 \pmod 4$, a necessary condition for existance of a symmetric BIBD with paramters s, $v = s^2 + s + 1 = b$, $r = s + 1 = k$, $\lambda = 1$ ~~the square free part of~~ is that if p is an odd prime dividing the square free part of s, then

$$p \not\equiv 3 \pmod 4.$$

proof :- let, u be the square free part of $r - \lambda = s$, then

$$1 = \left((-1)^{\frac{v-1}{2}} \cdot 1 \mid p \right)$$

$$= \left((-1)^{\frac{s(s+1)}{2}} \mid p \right)$$

$$= (-1 \mid p) = (-1)^{\frac{p-1}{2}}.$$

$\therefore \frac{p-1}{2}$ is even

$\Rightarrow p \not\equiv 3 \pmod 4.$

e.g. :- The above corollary suggests that the non-existence of BIBD of the series $v = s^2 + s + 1 = b$, $r = s + 1 = k$, $\lambda = 1$ and of the series, $v = s^2$, $b = s^2 + s$, $r = s + 1$, $k = s$, $\lambda = 1$ for $s = 6, 14, 21, 22, 30$ etc.

✓ Theorem :- 1. The necessary condition for existence of a BIBD can be stated as :-

$\lambda(v-1) \equiv 0 \pmod{(k-1)}$

$\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$

2. A necessary & sufficient condition for BIBD with v treatments with $k = 3, 4$ is that

$$\lambda(v-1) \equiv 0 \pmod{(k-1)}$$

and $$\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$$

3. A BIBD design with $k = 5$ exists for every λ satisfying two conditions;

$$\lambda(v-1) \equiv 0 \pmod{(k-1)}$$

$$\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$$

except for the non-existence design $v = 15$, $b = 21$, $r = 7$, $k = 5$, $\lambda = 2$

sufficient cond^n →

4. A BIBD design with $k = 6$ exists for $\lambda < v$ satisfying two

$$\lambda(v-1) \equiv 0 \pmod{(k-1)}$$

$$\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$$

except for the non-existence design

$v = 21$, $b = 28$, $r = 8$, $k = 6$, $\lambda = 2$.

5. A BIBD design with $k = 7$ exists for all v & $\lambda \equiv 0, 6, 7, 12, 18, 24, 30, 35, 36 \pmod{42}$

& for $\lambda > 30$, but not divisible by 2 or 3.

satisfying

$$\lambda(v-1) \equiv 0 \pmod{(k-1)}$$

$$\lambda v(v-1) \equiv 0 \pmod{k(k-1)}.$$

1. Show that, the following BIBD can be constructed for all integral values of $m \geq 2$

(i) $b = 2^{m-1} = v$, $r = 2^{m-1} = k$, $\lambda = 2^{m-2}$

(ii) $v = 2^{m-1}$, $b = 2(2^{m-1} - 1)$, $r = 2^{m-1} - 1$

$k = 2^{m-2}$, $\lambda = 2^{m-2} - 1$.

(iii) $v = 2^{m-1} - 1$, $b = 2(2^{m-1} - 1)$

$r = 2(2^{m-2} - 1)$, $k = 2^{m-2} - 1$,

$\lambda = 2(2^{m-3} - 1)$

(iv) $v = 2^{m-1} - 1$, $b = 2(2^{m-1} - 1)$,

$r = 2^{m-1}$, $k = 2^{m-2}$, $\lambda = 2^{m-2}$

2. Using finite geometrices construct the following BIBD

$v = 15$, $b = 35$, $r = 7$, $k = 3$, $\lambda = 1$

$v = 40$, $b = 40$, $r = 13$, $k = 13$, $\lambda = 4$

$v = 27$, $b = 39$, $r = 13$, $k = 9$, $\lambda = 4$

3. Show that, starting from the BIBD $v = 7 = b$, $r = 3 = k$, $\lambda = 2$, we get the Sol's of following BIBD

(i) $v = 16 = b$, $r = 6 = k$, $\lambda = 2$

(ii) $v = 16 = b$, $r = 10 = k$, $\lambda = 6$

(iii) $v = 10$, $b = 15$, $r = 6$, $k = 4$, $\lambda = 2$

(iv) $v = 6$, $b = 15$, $r = 5$, $k = 2$, $\lambda = 1$.

4. Show that the following BIBD does not exist for odd v

(i) $v = 137 = b$, $r = 17 = k$, $\lambda = 2$

(ii) $v = 53 = b$, $r = 13 = k$, $\lambda = 3$

(iii) $v = 103 = b$, $r = 18 = k$, $\lambda = 3$

5. Show that the following BIBD does not exist for even v

(i) $v = 29 = b$, $r = 7 = k$, $\lambda = 2$

(ii) $v = 46 = b$, $r = 10 = k$, $\lambda = 2$

(iii) $v = 106 = b$, $r = 21 = k$, $\lambda = 4$.

6. A $(v \times b)$ matrix D with entries $-1, 0, 1$ is said to be balanced orthogonal design (BOD) if

(i) the inner-product of any two rows are 0

(ii) when -1 is replaced by $+1$, the resultant matrix

becomes the incidence matrix of a BIBD with parameters v, b, r, k, λ, show that a necessary condition for existence of a BIBD is that λ is even.

11.05.2011

Further construction of BIBD:- Construction of BIBD of the series $b = v = 4\lambda + 3$, $r = k = 2\lambda + 1$, λ. If $4\lambda + 3$ is a prime or a prime power, then the initial block formed of the even power of the primitive root of $GF(4\lambda + 3)$ gives the above series of BIBD. The odd powers of the primitive roots also formed another initial block which gives the same design. It can be verified that, among $2\lambda(2\lambda + 1)$ differences between the even power of α, i.e., each of the non-zero elements of $GF(4\lambda+3)$ occurs λ times. When $(4\lambda + 3)$ is a prime power, the initial block is developed by adding interms each of the different non-zero elements of the field to the elements of the initial block.

e.g.:- The design, $v = b = 11$, $r = k = 5$, is obtained by developing the initial block formed of the even powers of 2 which the primitive root of GF(11). The initial block is then 1, 4, 5, 9, 3. The design is shown below

1	4	5	9	3
2	5	6	10	4
3	6	7	0	5
.	.	.	.	.
.	.	.	.	.
10	2	3	7	1
0	3	4	8	2

Construction of BIBD by developing initial block

Let, v treatments in a BIBD, is denoted by the elements of mod v. Let, $x_1, x_2, \ldots$ be contains of an initial block where $x_1, x_2, \ldots, x_k$ are the elements of modulo by taking all possible difference of two at a time of the k elements in the initial block, we get $k(k-1)$ such

differences. Let, among these differences each of the non-zero elements of mod v occurs a constant time, say λ, then by developing the initial block a symmetrical BIBD with parameters $v=b$, $k=r$, λ is obtained. By developing an initial block under modulo v, we mean getting other blocks from it. The 1st block is obtained by adding unity to each element of the initial block under modulo v and taking these sums as the contents of the block, repeating this process design is obtained.

e.g.:- Taking elements of mod 13, as the treatments, let

0 1 3 9 form an initial block.

The differences under modulo 13 of these 4 elements taking 2 at a time are as follows:

0	1	3	9
	1	3	9
	12	10	4
		2	8
		11	5
			6
			7

It is seen that among the 12 differences, each of the non-zero elements of module 13 occurs once. Hence, by developing this block modulo 13. A symmetrical BIBD design with parameters $v = b = 13$, $r = k = 4$, $\lambda = 1$ is obtained.

The concept behind this method:- If the difference between the 2 elements in the initial block is D, then when the block is developed to get other blocks, all possible pairs of elements which differ by D occur together in the different blocks. If there are λ pairs of elements, each giving a difference 1 then all possible pairs of nos differing by D occur together in the different blocks λ times.

Now, the initial block is so chosen that D takes each of the values from 1 to $(v-1)$ equally often.

Hence, all pairs of nos occur together equally often in the different blocks of the design.

Case of p initial blocks:- Instead of taking 1 initial block, p initial blocks can also be taken. From each of these blocks $k(k-1)$ differences are obtained if among the $pk(k-1)$ differences obtain from all the p blocks. Each of the non-zero elements of the concern field occurs say, λ times by developing these p blocks, a BIBD with the following parameters is obtained

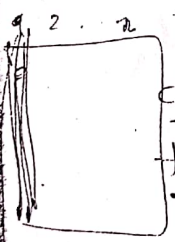
$v = v$, $b = pv$, k, $r = pk$, λ.

Dual design:- Let, $\theta_1, \theta_2, \ldots, \theta_v$ denotes the treatments in a BIBD with b blocks. Let the blocks be numbered by $1, 2, 3, \ldots, b$. Any treatment θ_i, $i = 1(1)v$ occurs in r blocks. Let, a block β_i be form with these r block nos, then the design with the blocks $\beta_1, \beta_2, \ldots, \beta_v$ is called the dual design of the original BIBD. The parameters of the designs are

$v' = b$, $b' = v$, $r' = k$, $k' = r$. The resultant design is not however BIBD always. If the original design in the symmetrical BIBD, then the dual design be also a symmetrical design with same parameters.

Method of symmetrically repeated differences: 12.05.2011

Consider a module containing n elements. To each element of the module, there correspond m treatments, the treatments corresponding to the elements being denoted by $a_1, a_2, \ldots, a_m$. The treatment a_i is said to belongs to the ith class. Thus, we have mn treatments belonging to each of m classes. With any ordered pair of distinct treatments a_i & b_j, we associate the difference $(a-b)$ of the type $[i,j]$. Each difference is an element of n & is of a certain type. If $i=j$, the difference is said to be a Pure difference. Obviously, in this case $a \neq b$ as the treatments are distinct. If $i \neq j$, the difference is said to be mixed difference.



Suppose, a block B containing k distinct treatments is given from this block, and can get $k(k-1)$ differences. These differences are called differences arrising out of the block B.

e.g.: Consider the module M consisting of residue classes modulo 5 and let to each element of a n, there are 2 treatments a_1, a_2, then the differences associated with the ordered pair of treatments. 2_1 and 3_1 is a pure difference of type $[1,1]$ while the difference associated with the pair of treatments $3_2, 4_1$ is the mixed difference of 4 of type $[2,1]$.

Suppose, B is the block $2_1, 4_2, 0_2$, the differences arrising out of this blocks are pure differences of 4, 1 are the type $[2,2]$

and the mixed differences 3 & 2 of the type $[1,2]$ and mixed differences 2 & 3 of the type $[2,1]$.

Since, there are m classes & the no. of elements in M is n, there are $(n-1)$ pure differences of the type $[i, i]$ for $i = 1, 2, \ldots, m$. Similarly, there are n mixed differences of the type $[i,j]$ for $i,j = 1, 2, \ldots, m$, $i \neq j$.

Now, consider a set of t blocks $b_1, b_2, \ldots, b_t$. If among the differences arrising from these t blocks, each possible difference occure a cont no. of times, say λ times. The differences are said to be symmetrically repeated, λ times each.

{0,1,2,...,6}

{0 1 2 3 4}

e.g.: Let, M be the module consisting of residue classes of mod 5. Let, there corresponds 3 treatments, A as $\alpha_1, \alpha_2, \alpha_3$. Consider the following 7 blocks.

0 1 2 3 4 5
0 1 2 3 4 0

$\{(0_1, 1_1, 2_1); (0_2, 1_2, 2_3); (0_3, 1_3, 2_1); (0_1, 2_1, 3_2)$
$(0_2, 2_2, 0_3); (0_3, 2_3, 0_1); (0_1, 2_2, 1_3)\}$.

Differences arrising out of these blocks.

(0 123 4:

0_1
0_2
0_3

M.

Blocks	Differences of the type							
	[1,1]	[2,2]	[3,3]	[1,2]	[1,3]	[2,1]	[2,3]	[3,1]	[3,2]
$(0_1, 1_1, 0_2)$	1,1	-	-	0,1	-	0,1	-	-	
$(0_2, 1_2, 2_3)$	-	-	-	-	-	-	3,1	-	2,1
$(0_3, 1_3, 2_1)$	-	-	1,1	-	2,1	-	-	3,4	-
$(0_1, 2_1, 3_2)$	3,2	-	-	2,4	-	3,1	-	-	-
$(0_2, 2_2, 0_3)$	-	3,2	-	-	-	-	0,2	-	0,3
$(0_3, 2_3, 0_1)$	-	-	3,2	-	-	0,3	-	-	0,2
$(0_1, 2_2, 1_3)$	-	-	3	4	2	1	1	1	

Pure diff = {1,2,3,4}.
paired = {0, 1, 2, 3, 4}.

Table shows that among the differences arising out of the 7 blocks. Each difference is repeated symmetrically. It must be noted that 0 can not occure as pure difference as the block contents are distinct. Here, $\lambda = 1$.

Bose's 1st Fundamental theorem on method of differe

Let, M be a module containing n elements $x^{(0)}, x^{(1)}, \ldots, x^{(n-1)}$ and to each element $x^{(u)}$, let there correspond m treatments $x_1^{(u)}, x_2^{(u)}, \ldots, x_m^{(u)}$, $u = 0, 1, \ldots, n-1$.

Thus, there are mn treatments. The treatment $x_i^{(u)}$ is said to belong to the i-th class given any block B_s containing k distinct of these treatments. One can obtain n blocks $B_{s,\theta}$, where θ range over the elements of m in the following manner. Corresponding to any treatments $x_i^{(u)}$ of the i-th class occuring in B_s. We take the treatment $x_i^{(v)}$ of the ith class in $B_{s,\theta}$ where, $x_i^{(v)} = x_i^{(u)} + \theta$. Then, n blocks $B_{s,\theta}$ are said to be obtained

by developing the block B_s. Clearly, $B_{s,0} = B_s$.

Now, consider the set of treatments. Let it be possible to find a set of t blocks $B_1, B_2, \ldots, B_t$ satisfying the following conditions :-

(1) Each block contained k distinct treatment

(2) Among the kt treatments occuring the t blocks exactly r treatments belongs to each of classes.

(3) The differences arising from the t blocks are symmetrically repeated λ times in each.

Then the nt blocks obtained by developing the initial blocks $B_1, B_2, \ldots, B_t$ provide us with a solution of BIBD with parameters $v = mn$, $b = nt$, r, k, λ.

Eg:- 1 Let M be the module consisting of residue classes of mod 7 and let to each element of M, there correspond just 1 treatment. Thus, the treat numbered 0, 1, .., 6. Consider, the initial block 0, 3, 5, 6. It can be easily verified that (B.T.) among the differences arrising out of this initial block every non-zero element of M appears twice. Thus, $\lambda = 2$. Developing initial block, we get a BIBD with parameters $v = 7 = b$, $r = 4 = k$, $\lambda = 2$.

2. Consider a module M consisting of residue classes mod 5 and to each element a of M. Let there correspond 3 treatments a_1, a_2, a_3. Consider the initial blocks

$\{(0_1, 1_1, 0_2), (0_2, 1_2, 2_3), (0_3, 1_3, 2_1),$
$(0_1, 2_1, 3_1), (0_2, 2_2, 0_3), (0_3, 2_3, 0_1),$
$(0_1, 2_2, 1_3)\}$.

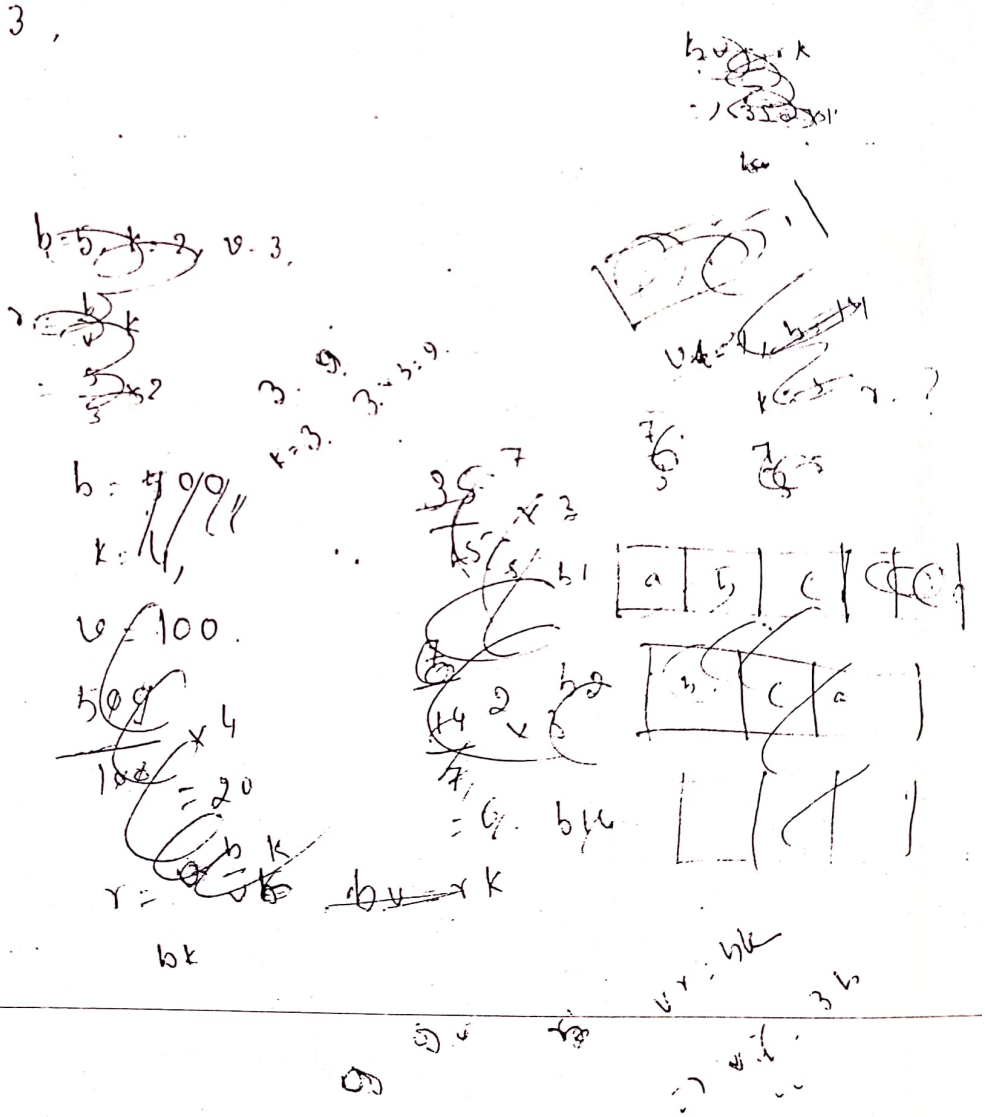
We have already seen that the differences arising out of the initial blocks are symmetrically

repeated with $\lambda = 1$. Also, cond^{n} 1 & 2 of the theorem are satisfied. Thus, the 7 initial blocks when developed

given the ~~initial block~~ BIBD with parameters

$v = 15$, $r = 7$, $b = 35$

$k = 3$,



Fundamental Theorem:- Let, M be a module consisting of n elements. $x^{(0)}_{(1)}, x^{(0)}_{(n-1)}, x^{(0)}, x^{(1)}, \ldots, x^{(n-1)}$ and to each element $x^{(u)}$, let there correspond t treatments $x^{(u)}_1, x^{(u)}_2, \ldots x^{(u)}_t$, $u = 0(1)n-1$. To the n treats, we adjoin a new treat ∞ (called the invariant treatment), so that we have $v = Mn+1$ treatments. Given, any block B_s containing k of these treatments, we can obtain from it n blocks $B_{s,\theta}$, where θ ranges over the elements of M in the following manner: corresponding to any treatments $x^{(u)}_i$ of the ith class occurring in B_s, we take the treat $x^{(v)}_i$ of the ith class in $B_{s,\theta}$, where $x^{(v)} = x^{(u)} + \theta$. If ∞ occurs in B_s, then ∞ also appears in $B_{s,\theta}$. The n blocks $B_{s,\theta}$ are said to be obtained by developing B_s. Now, consider the n treatments together with the invariant treatment ∞. Let, it be possible to find a set of $(t+u)$ blocks, $B_1, B_2, \ldots, B_t, B'_1, B'_2, \ldots, B'_u$ satisfying the following conditions.

(1) Each of the blocks $B_1, B_2, \ldots, B_t$ contains k distinct treatments $x^{(u)}_i$ where as each of the blocks $B'_1, B'_2, \ldots, B'_u$

contains α and $(k-1)$ distinct treatments $x_i^{(a)}$.

(2) Among the kt treatments occurring in the blocks B_i, exactly $(nu-\lambda)$ belongs to each of the m classes, whereas among the $u(k-1)$ treatments occurring in the blocks B_i', exactly λ belong to the each of the m classes.

(3) The differences arrising from the $(t+u)$ blocks $B_1, B_2, \ldots, B_t, B_1'', \ldots, B_u''$ are symmetrically repeated λ times. Where the blocks B_j'' are obtained from B_j' by deleting the treatment α, for $j = 1, 2, \ldots, u$.

Then the $n(t+u)$ blocks, obtained by developing the initial block B_i & B_i' provide a solution for a BIBD with a parameters $v = mn+1$, $b = n(t+u)$, $r = nu$, $k = k, \lambda$.

$v = mn+1$,

Eg:- BIBD with $k=3$, $\lambda=1$. A steiner's triple system is an arrangement of v objects in triplet s.t. every pair of objects appears exactly once in each treatment. A steiner's triple system is an BIBD with v $k=3$, $\lambda=1$. Now, we have $b = \frac{v(v-1)}{6}$, $r = \frac{v-1}{2}$

Therefore, $v = 2r+1$, $b = r(2r+1)$. Hence, r must be of the form $3t+1, 3t$ We get the following 2 series of steiner's triple system

S_{11}: $v = 6t+3$, $b = (3t+1)(2t+1)$

$r = 3t+1$, $k = 3$, $\lambda = 1$

S_{12}: $v = 6t+1$, $b = t(6t+1)$

$r = 3t$, $k = 3$, $\lambda = 1$,

Consider the 1st series, S_{11}. Let M be the module of the residue classes mod $(2t+1)$ and let its elements be $0, 1, \ldots, 2p$. To any element $u \in M$, get there corresponds 3 treatments u_1, u_2, u_3. Show that there are $6t+3$ treatments in all. Consider, the pairs

$[1, 2t], [2, 2t-1], \ldots, [t, t+1]$ ... (*)

The differences arising from the ith pair $[i, 2t+1-i]$ are $(2t+1-2i)$ and $2i$.

Taking $i = 1, 2, \ldots, t$, we get all the non-zero elements of M.

Now, consider the following set of initial blocks

$[1_1, (2t)_1, 0_2]; [2_1, (2t-1)_1, 0_2], \ldots$

$[t_1, (t+1)_1, 0_2];$

$[1_2, (2t)_2, 0_3]; [2_2, (2t-1)_2, 0_3] \ldots [t_2, (t+1)_2, 0_3];$

$[1_3, (2t)_3, 0_1]; [2_3, (2t-1)_3, 0_1]; \ldots [t_3, (t+1)_3, 0_1];$

$[0_1, 0_2, 0_3] \ldots$ (**)

It is clear that the pure differences of the type [1,1], [2,2], [3,3] arrising from (**) are repeated just one. Again among the pairs (*) every non-zero element of M occurs once. Hence, among the mixed differences of the type [1,2] arrising from the 1st row of (**), every non-zero element of M occurs just once. The mixed difference of the type [1,2] arrising from the last block in (**). No mixed difference of type [1,2] can occure from the 2nd or 3rd rows of (**). Hence, every elements of m occurs exactly once of the mixed difference of type [1,2] arising from (**). Similarly, we can prove that, the same thing for the mixed diff. of other types.

(T-A) ⊛

Theorem:- The initial block (4*) provide a solution for the series S_1 of BIBD with parameters $v = 6t+3$, $b = (3t+1)(2t+1)$, $r = (3t+1)$, $k = 3$, $\lambda = 1$.

e.g.:- Let, $t = 2$, so

$v = 15$, $r = 7$, $k = 3$, $\lambda = 1$.

$b = 35$,

The initial block in this case are

$(1_1, 4_1, 0_2)$, $(1_2, 4_2, 0_3)$, $(1_3, 4_3, 0_1)$, $(2_1, 3_1, 0_2)$, $(2_2, 3_2, 0_3)$, $(2_3, 3_3, 0_1)$, $(0_1, 0_2, 0_3)$.

The full design can be obtained by developing these initial blocks.

e.g.:- Let, $t = 1$, then

$v = 9$, $b = 12$, $r = 4$, $k = 3$, $\lambda = 1$.

Here, $2t+1 = 3$, the initial blocks are

$(1_1, 2_1, 0_2)$; $(1_2, 2_2, 0_3)$; $(1_3, 2_3, 0_1)$; $(0_1, 0_2, 0_3)$.

The complete design has the following 12 blocks.

$(1_1, ?_1, 0_2)$; $(?_1, 0_1, 1_2)$; $(0_1, 1_1, ?_2)$; $(1_2, ?_2, 0_3)$;
$(?_2, 0_2, 1_3)$; $(0_2, 1_2, ?_3)$; $(1_3, ?_3, 0_1)$;
$(?_3, 0_3, 1_1)$; $(0_3, 1_3, ?_1)$; $(0_1, 0_2, 0_3)$;
$(1_1, 1_2, 1_3)$; $(?_1, ?_2, ?_3)$.

Theorem:- There exists a soln of a series S_{12} of a BIBD when $v = 6t+1$ is a prime or prime power.

Proof:-

e.g.:- Let, $t = 2$, then $v = 13$, $b = 26$, $r = 6$, $k = 3$, $\lambda = 1$. primitive element of $GF(13)$ is 2. Thus, the initial blocks are $(1, 3, 9)$, $(?, 6, 5)$.

Hadamard Matrix:- A square matrix of order n and entries -1 & 1 is said to be a Hadamard matrix (H).

If $HH^T = nI_n$ then $H^TH = nI_n$. Also, it follows that, if any row (or column) of H is multiplied by -1, the matrix remain same of H.

A necessary condn for the existence of H matrix of order n

is that $n \equiv 0 \pmod 4$ with n - 1 & 2 being the trivial cases.

It is ~~known~~ not known whether this condn. is sufficient or not. However, Hadamard ~~Hardamard~~ for all permissible ~~perpusiv. able~~ values of $n \leq 200$ are known to exists.

Since, a Hadamard # matrix remains Hadamard # matrix, when any rows (or columns) is multiplied by -1, it is always possible to write a Hadamard # matrix with its 1st row and the 1st column containing 1 only. This Hadamard # matrix is said to be in its normal form.

e.g.:- $H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $H_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$

Let, H be a Hadamard matrix of order $n = 4t$ in its normal form and B be a matrix obtained from H by deleting its 1st row & column of all unities, then B is a square matrix of order $4t - 1$.

Let, $N = \frac{BB'}{2}$ $\begin{pmatrix} \cdots \\ \cdots \end{pmatrix}$ $\begin{pmatrix} \cdots \\ \cdots \end{pmatrix}$

where J is a $(4t-1)$ components column vector of all unities. i.e. N is a matrix which can be construct from B just change all -1 by 0, then N is the incidence matrix of a BIBD with parameters $v = 4t-1 = b$, $r = 2t-1 = k$, $\lambda = t-1$.

Conversely, if N is the incidence matrix of a BIBD with parameters as a row, then changing the 0 in N by -1 and annexing a row & column of 1, the resultant is a Hadamard matrix of order $4t$.

e.g.: $v = 15 = b$, $r = 7 = k$, $\lambda = 3$

Since, 15 is not a prime, then method of difference does not exist.

So, there exists a Hadamard matrix

$$H_{16} = \begin{pmatrix} H_4 & H_4 & H_4 & H_4 \\ H_4 & -H_4 & H_4 & -H_4 \\ H_4 & H_4 & -H_4 & -H_4 \\ H_4 & -H_4 & -H_4 & H_4 \end{pmatrix}$$

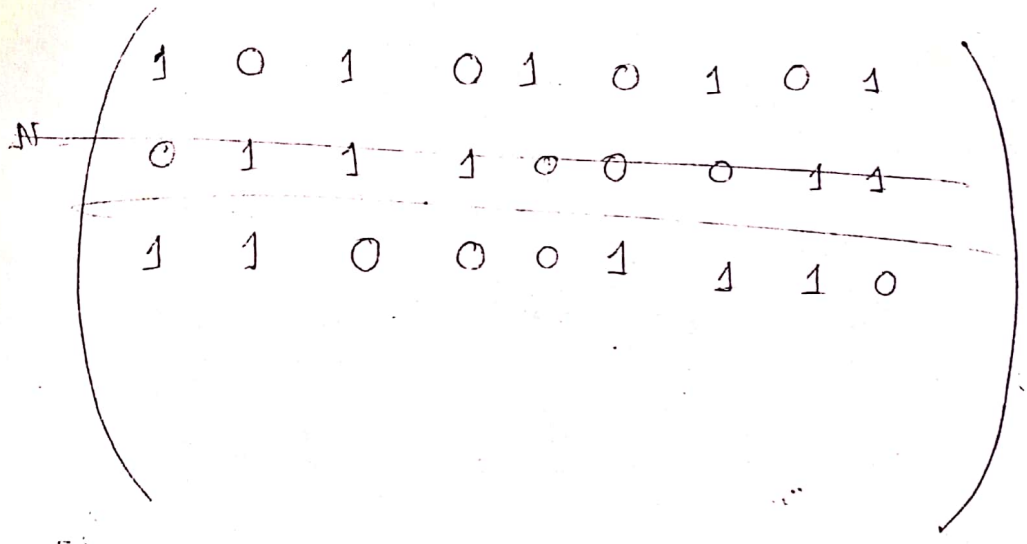
(b)

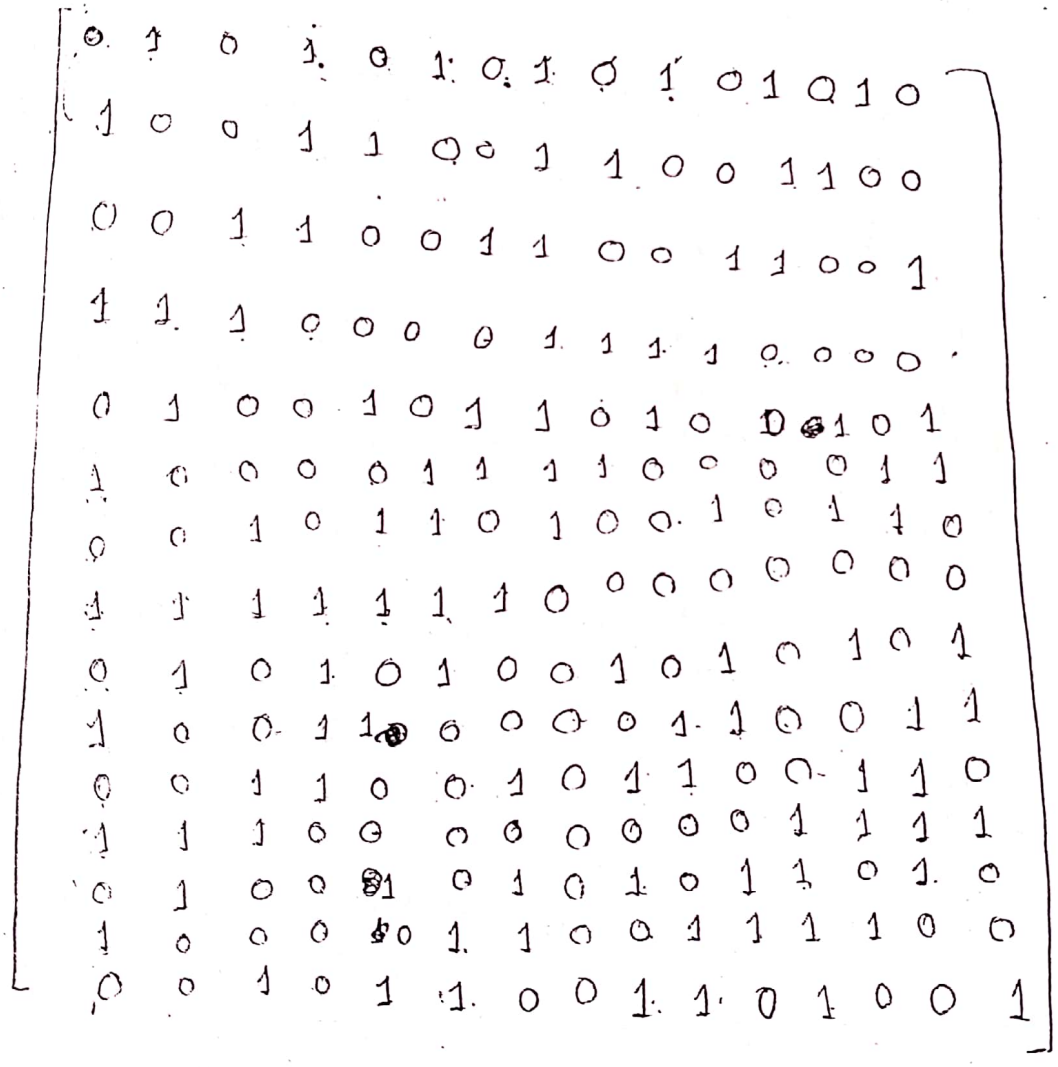
Removing the 1st row & 1st column and changing -1 to 0, H16, we get a incidence matrix of a BIBD with $v = 15 = b$, $r = 7 = k$, $\lambda = 3$.

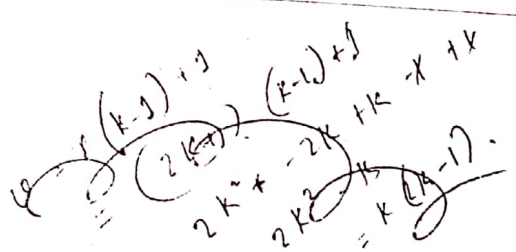
15 x 15

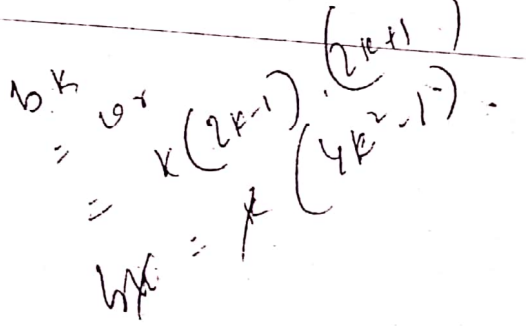
$$N = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$









Theorem:- If there exists a BIBD with $r = 2k+1$, $\lambda = 1$, then we can construct a BIBD with parameter $v^* = 4k^2 - 1 = b^*$, $r^* = 2k^2 = k^*$, $\lambda^* = k^2$.

Proof:- Let D be BIBD with parameters $r = 2k+1$, $\lambda = 1$.

Parameters of D are thus

$v = k(2k-1)$, $b = 4k^2 - 1$, $r = 2k+1$, k, $\lambda = 1$. [$\because \lambda(v-1) = r(k-1) \Leftrightarrow 1 \times (v-1) = (2k+1)(k-1) \Leftrightarrow v - k(2k-1)$]

$\because vr = bk \Leftrightarrow b = 4k^2 - 1$]

Let, N be the incidence matrix of D, then,

$$NN' = 2kI + JJ'$$

Further, since, $\lambda = 1$, it can be seen that $N'N = kJ + M$.

where M is a square matrix of order $b = 4k^2 - 1$.

Also, M is a symmetric matrix with zero and unity as its entries.

Further, $MJ = (N'N - kI)J = 2k^2 J$

and $J'M = 2k^2 J'$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Finally, $MM' = k^2I + k^2JJ'$.

Thus, M is a incidence matrix of symmetric BIBD with required parameters.

Theorem:- Suppose N is the incidence matrix of a BIBD with parameters $v^* = 4t-1 = b^*$, $r^* = 2t-1 = k^*$, $\lambda^* = t-1$ and $\bar{N}$ be the incidence matrix of complementary design, then M is given by

$$M = \begin{bmatrix} N & \bar{N} & 0 \\ J' & 0' & 0 \\ N & N & J \end{bmatrix}$$ is the incidence matrix of a BIBD with parameters $v = 8t-1 = b$, $r = 4t-1 = k$, $\lambda = 2t-1$.

Proof:- (H.T.).

20.05.20

Optimality :-

Block design :- The block design have following parameters

v : No. of treatments,

b : " " blocks ,

k_j : " " plot (experimental unit) in jth block, $j = 1(1)b$

r_i : " " replications of ith treatment, $i = 1(1)v$

The incidence matrix N of the design is

$N = N_{ij}(v \times b) \geq 0$ is the no. of times the ith treatment appears in jth block ;

$\underset{\sim}{k} = (k_1, k_2, \ldots, k_b)^T$, the column vector of block sizes

$\underset{\sim}{r} = (r_1, r_2, \ldots, r_v)^T$, the column vector of replications

$K = \text{Diag}(k_1, k_2, \ldots, k_b)$

$R = \text{Diag}(r_1, r_2, \ldots, r_v)$

Intra block analysis of block design :- Suppose, D is a block design with parameters $v, b, \underset{\sim}{r}, \underset{\sim}{k}$ N. Let, y_{iju} is the uth observation pertaining to the ith treatment in the jth block. $\underset{\sim}{y}$ be the vector of all observations (y_{iju}). There are n observations. Also, let, D_1 be the $v \times n$ (treatment Vs. observation) incidence matrix i.e. if $D_1 = ((d_{rs}))$

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Then,

(d_{rs}) = 1 if sth observation comes from rth treatment.

= 0 if s" " does not come " r" "

Similarly, D_2 be the $(b \times n)$ blocks vs. observations incidence matrix.

Finally, let T_i, $i = 1(1)v$ denote the total of the observations from the ith treatment. B_j, $j = 1(1)b$, the total of the observations of jth blocks and

$G = \sum_{i=1}^{v} T_i$, grand total.

(11.7) Checking

1. $R = D_1 D_1^T$

2. $k = D_2 D_2^T$

3. $N = D_1 D_2^T$

4. $T = (T_1, T_2, \ldots, T_v)' = D_1 \underset{\sim}{Y}$

5. $B = (B_1, B_2, \ldots, B_b)' = D_2 \underset{\sim}{Y}$.

6. $\underset{\sim}{r} = D_1 J_n = R J_v$

7. $\underset{\sim}{k} = D_2 J_n = k J_b$.

Fixed effect model for the design:

~~$\underset{\sim}{Y} = \mu \underset{\sim}{J} N + D'$~~

$$\underset{\sim}{Y} = \mu J_n + D_1' \underset{\sim}{T} + D_2' \underset{\sim}{\beta} + \underset{\sim}{\epsilon}$$

where, μ is the general mean

$T = (T_1, T_2, \ldots, T_v)'$: Vector of treatment effects

$\beta = (\beta_1, \beta_2, \ldots, \beta_b)'$: vector of block effects.

$\underset{\sim}{e}$ = Error of estimation

Applying the l.s.e. method, we get

$\frac{\partial}{\partial \mu}(\underset{\sim}{e}'\underset{\sim}{e}) = 0$

$$\Rightarrow \frac{\partial}{\partial \mu}\left(\underset{\sim}{Y} - \mu J_n - D_1' \underset{\sim}{T} - D_2' \underset{\sim}{\beta}\right)'\left(\underset{\sim}{Y} - \mu J_n - D_1' \underset{\sim}{T} - D_2' \underset{\sim}{\beta}\right) = 0$$

$$\Rightarrow -\underset{\sim}{Y}' J_n - J_n' \underset{\sim}{Y} + 2\mu J_n' J_n + J_n' D_1' \underset{\sim}{T} + J_n' D_2' \underset{\sim}{\beta} + \underset{\sim}{T}' D_1 J_n + \underset{\sim}{\beta}' D_2 J_n = 0$$

$$\Rightarrow -2G + 2n\mu + 2J_b k \underset{\sim}{\beta} + 2J_v R \underset{\sim}{T} = 0$$

$$\Rightarrow n\mu + J_b' k \underset{\sim}{\beta} + J_v' R \underset{\sim}{T} = G$$

$$\frac{\partial}{\partial \gamma_i}\left(\underset{\sim}{e}'\underset{\sim}{e}\right) = 0$$

$$\Rightarrow -\underset{\sim}{y}' d_{ji}' + \mu J_n' d_{ji}' + r_i \gamma_i + n_i \beta_i = 0$$

$$\Rightarrow \mu J_n' d_{ji}' + \underset{\sim}{\beta}' \underset{\sim}{n_i}' + \alpha r_i \gamma_i = y' d_{ji}'$$

$(i = 1, 2, \ldots, v)$.

where d_{ji}' denotes the ith column of D_1' & $\underset{\sim}{n_i}'$ denotes the ith column of N'.

$$\therefore \mu J_1' D_1' + (N\underset{\sim}{\beta})' + (R\underset{\sim}{\gamma})' = y_1' D_1$$

$$\therefore \mu D_1 J_n + N\underset{\sim}{\beta} + R\underset{\sim}{\gamma} = D_1 \underset{\sim}{y}$$

$$\therefore k J_b \mu + N\underset{\sim}{\beta} + R\underset{\sim}{\gamma} = T$$

Similarly, from

$$\frac{\partial}{\partial \beta_i}\left(\underset{\sim}{e}'\underset{\sim}{e}\right) = 0, \text{ we get}$$

$$k J_b \mu + k\underset{\sim}{\beta} + N'\gamma = B.$$

Hence, the normal equns are

$$\begin{bmatrix} n & J_b' k & J_v' R \\ k J_b & k & N' \\ R J_v & N & R \end{bmatrix} \begin{bmatrix} \mu \\ \underset{\sim}{\beta} \\ \underset{\sim}{\gamma} \end{bmatrix} = \begin{bmatrix} G \\ B \\ T \end{bmatrix} \quad \text{---(*)}$$

Now

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Check :-

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$\therefore R_c$

Now, consider 2 non-singular ~~matrix~~ matrices A

~~are given by~~ are given by

$$A = \begin{bmatrix} I_b & 0 \\ -Nk^{-1} & I_v \end{bmatrix}, \quad B = \begin{bmatrix} I_b & 0 \\ -R^{-1}N & I_v \end{bmatrix}$$

Check :- (i) $n = \sum r_i = \sum k_j = J_v' \underset{\sim}{r} = J_b' \underset{\sim}{R}$

$\underset{\sim}{r}' = J_v' R$

$\underset{\sim}{k}' = J_b' K$

$\underset{\sim}{r} = R J_v$

$\underset{\sim}{R} = K J_b$

$N J_b = \underset{\sim}{r} = R J_v$

$J_v' N = \underset{\sim}{R}' = J_b' K.$

Now, $J_b' \begin{bmatrix} K J_b & K & N' \end{bmatrix} = J_v' \begin{bmatrix} R J_v & N & R \end{bmatrix}$

$= \begin{bmatrix} n & J_b' K & J_v' R \end{bmatrix} = \begin{bmatrix} n & \underset{\sim}{k}' & \underset{\sim}{r}' \end{bmatrix}.$

and $\begin{bmatrix} K \\ N \end{bmatrix} J_b = \begin{bmatrix} N' \\ R \end{bmatrix} \quad J_v = \begin{bmatrix} K & J_b \\ R & J_v \end{bmatrix}$

$\therefore \text{Rank} \begin{bmatrix} n & J_b' K & J_v' R \\ K J_b & K & N' \\ R J_v & N & R \end{bmatrix} = \text{Rank}(E).$

where $E = \begin{bmatrix} K & N' \\ N & R \end{bmatrix}$

$$AE = \begin{bmatrix} I_b & 0 \\ -NK^{-1} & I_v \end{bmatrix} \begin{bmatrix} K & N' \\ N & R \end{bmatrix} = \begin{bmatrix} K & N' \\ 0 & C \end{bmatrix}$$

where $C = R - NK^{-1}N'$.

$$\text{and } EB = \begin{bmatrix} K & N' \\ N & R \end{bmatrix} \begin{bmatrix} I_b & 0 \\ -R^{-1}N & I_v \end{bmatrix} = \begin{bmatrix} D & N' \\ 0 & R \end{bmatrix}$$

where $D = K - N'R^{-1}N$.

Hence, $\text{Rank}(E) = \text{Rank}(AE) = \text{Rank}(C) + b$.

$\therefore \text{Rank}(E) = \text{Rank}(EB) = \text{Rank}(D) + v$.

$\therefore \text{Rank}(C) + b = \text{Rank}(D) + v$.

Now, $J_v'C = J_v'R - J_v'NK^{-1}N'$

$= \underset{\sim}{r}' - J_b'KK^{-1}N'$

$= \underset{\sim}{r}' - J_b'N'$

$= \underset{\sim}{r}' - \underset{\sim}{r}' = \underset{\sim}{0}'$

(Check) $CJ_v = \underset{\sim}{0}$

$J_b'D = \underset{\sim}{0}'$

$DJ_b' = \underset{\sim}{0}$

Hence, C & D are symmetric Doubly centre matrices i.e. row & col sum = 0

Now, we

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Now, premultiplying both sides of $\begin{bmatrix} 0 & \cdots & N'k \end{bmatrix}$ we get

$$\begin{bmatrix} -NJ_b + RJ_v & \cdot N'N & \cdot Nk^{-1}N'R \end{bmatrix} \begin{bmatrix} \mu \\ \beta \\ \tau \end{bmatrix}$$

where $Q = T - Nk^{-1}B$

$$\Rightarrow C\underset{\sim}{\tau} = Q.$$

The vector Q is called the vector of adjusted treatment total and the above equations are called Reduced intrablock normal equations for treatment effects.

$C J_v = 0$, it follows C is a singular matrix.

$$R(C) \leq v - 1.$$

Further, if $\underset{\sim}{p}^T \underset{\sim}{\tau}$ is a linear combination of treatment effects, then a necessary condition for $\underset{\sim}{p}^T \underset{\sim}{\tau}$ is estimable is that

$\underset{\sim}{p}^T J = 0$ i.e. $\underset{\sim}{p}^T$ is a contrast.

A parametric fn $\underset{\sim}{p}^T \underset{\sim}{\tau}$ is called a contrast of treatment effect if $\underset{\sim}{p}^T J = 0$.

A treatment constrast $\underset{\sim}{p}^T \underset{\sim}{\tau}$ is called

an elementary contrast if the two ~~to~~ non-zero entries 1 & -1 and other entries are 0.

Def:- A Block design is said to be connected if all elementary ~~treatment~~ treatment contrasts are estimable.

Theorem:- A block design is connected iff $R(C) = v-1$.

25.05.2011

Proof:- Let, the design D be connected.

Consider, a set of $v-1$ LIN treatment contrast $\tau_1 - \tau_j$, let, this contrast be denoted by $c_2^T\tau, c_3^T\tau, \ldots, c_v^T\tau$, $j = 2(1)v$

where $\tau = (\tau_1, \tau_2, \ldots \tau_v)'$.

Obviously, the vectors $c_2, c_3, \ldots, c_v$ form the basis of a vector space of dimension $(v-1)$. Thus, any contrast $p^T\tau$ is expressible as a linear combination of the contrast $c_i^T\tau$, $i = 2(1)v$.

Also, $p^T\tau$ is estimable iff $p \in$ Col sp. of matrix C of design D.

Therefore, the dim of col sp of C must be same as that of the vector space spanned by the vectors c_i, $i = 2, 3, \ldots, v$;

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i.e. $\rho(v_0 - 1)$, it follows that

$$R(C) = v - 1.$$

Conversely, let $R(C) = v - 1$.

Let, $\xi_1, \xi_2, \ldots, \xi_{v-1}$ be a set of orthonormal eigen vectors corresponding to the non-zero eigen values $\theta_1, \ldots, \theta_{v-1}$ of C.

Note that, θ_i's may not be distinct.

Then,

$$E(\xi_i' Q) = E(\xi_i' C \hat{\tau}) = E(\theta_i \xi_i' \hat{\tau})$$
$$= \theta_i \xi_i' \tau.$$

Thus, an u.e. of $\xi_i' \tau$ is $\frac{\xi_i' Q}{\theta_i}$. Also, since, each ξ_i is orthogonal to J_v because $\theta_i \xi_i' J_v = \xi_i' C J_v = 0$, and ξ_i's are mutually $\perp$ (or

$\because E(\xi_i' Q) = \theta_i \xi_i'$

$\Rightarrow E\left(\frac{Q}{\theta_i}\right) = \tau$

any contrast $p'\tau$ belongs to the vector space spanned by $\{\xi_i \mid i = 1, 2, \ldots, v-1\}$

i.e. $$p = \sum_{i=1}^{v-1} a_i \xi_i$$

Now, $$E\left(\sum_{i=1}^{v-1} a_i \xi_i' Q / \theta_i\right) = p' \tau.$$

Thus, $p'\tau$ is estimable and hence the proof.]

Alternative def:: of connectedness :-

A treatment i and a block j in a block design are said to be associated if the treatment i appears in the block j. 2 treatments are said to be connected if it's possible to pass one to the other through a ~~chain~~ chain consisting alternatively of treatment & block such that any 2 consequtive member of a chain is associated. A design is said to be connected if every pair of treatment is connected.

Theorem :- Two def:: of connectedness is equivalent.

proof :- Suppose, the design is connected as for the 2nd def::.

Let, y_{ij} be the yield of any unit in the j^{th} block containing the i^{th} treatment.

Suppose, the treatment (i, i_1) are connected in the j_1^{th} block,

(i_1, i_2) in the j_2^{th} block, ...

(i_m, i') in the j_{m+1} th block.

Consider, the linear function

$$l = y_{ij} - y_{ij_1} + y_{ij_2} - y_{i_2 j_2} + \cdots$$

$$+ y_{i_m j_{m+1}} - y_{i' j_{m+1}}$$

Now, $E(l) = \tau_i - \tau_{i'}$

Hence $\tau_i - \tau_{i'}$ is estimable.

i.e. the design is connected as for 1st defn.

Conversely, let $R(C) = v-1$ if the 1st row of $C = (c_{ij})$ contains all non-zero elements, then the design is connected as for 2nd defn.

Suppose, in the last row of C the first m elements $(1 \le m \le v-2)$ are 0 and rest are non-zero.

[Note that, last row cannot be 0 and $m \ge v-1$

$\Rightarrow m = v$ as row sum and col sum of C is 0]

Then, all the elements c_{ij}, $m+1 \le i \le v$, $1 \le j \le m$ cannot be zero for then, $R(C) \le v-1$ (because $R_1^T = \sum R_i$, $C_v^T = \sum C_j$ gives first row 0, and last col 0 and then last $v-m$ row is indpt). ~~If for each~~

If for each j, $1 \le j \le m$ at least one c_{ij} is non-zero, $m+1 \le i \le v$, then treat $1, 2, \ldots, v$ are connected.

Let us, suppose, therefore that $c_{ij} = 0$, $m+1 \le i \le$ $1 \le j \le s$ and for each j, $j = s+1, s+2, \ldots, m$, atleast one $c_{ij} \neq 0$, for $i = m+1, m+2, \ldots, v$.

The treat $s+1, s+2, \ldots v$ are connected $\Leftarrow$ 1st m rows of C are

$$\begin{bmatrix} A_1 & A_2 & A_3 \\ A_2' & A_4 & A_5 \\ A_3' & A_5' & A_6 \\ 0 & 0 & A_7 \end{bmatrix}$$

where $A_1 = \begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1r} \\ c_{21} & c_{22} & \ldots & c_{2r} \\ \vdots & & & \\ c_{r1} & c_{r2} & \ldots & c_{rr} \end{bmatrix}$

A_3.

A_4

A_5 =

$$A_2 = \begin{bmatrix} c_{1,r+1} & \cdots & c_{1s} \\ c_{2,r+1} & \cdots & c_{2s} \\ \vdots & & \\ c_{r,r+1} & \cdots & c_{rs} \end{bmatrix}$$

$$A_3 = \begin{bmatrix} c_{1,s+1} & \cdots & c_{1m} \\ c_{2,s+1} & \cdots & c_{2m} \\ \vdots & & \\ c_{r,s+1} & \cdots & c_{rm} \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_{r+1,r+1} & \cdots & c_{r+1,s} \\ c_{r+2,r+1} & \cdots & c_{r+2,s} \\ \vdots & & \\ c_{s,r+1} & \cdots & c_{ss} \end{bmatrix}$$

$$A_5 = \begin{bmatrix} c_{r+1,s+1} & \cdots & c_{r+1,m} \\ c_{r+2,s+1} & \cdots & c_{r+2,m} \\ \vdots & & \\ c_{s,s+1} & \cdots & c_{sm} \end{bmatrix}$$

$$A_6 = \begin{bmatrix} c_{s+1,s+1} & \cdots & c_{s+1,m} \\ c_{s+2,s+1} & \cdots & c_{s+2,m} \\ \vdots & & \vdots \\ c_{m,s+1} & \cdots & c_{m,m} \end{bmatrix}$$

and A_7 is a $(v-m) \times (m-s)$ matrix such that each column of this matrix contains at least one non-zero element.

Now, $\begin{bmatrix} A_3 \\ A_5 \end{bmatrix} \neq 0$, for then $\text{Rank}(C) < v-1$.

If, every row of $\begin{bmatrix} A_3 \\ A_5 \end{bmatrix}$ contains at-least one non-zero element, treatments $1, 2, \ldots, v$ are connected. If not, let $A_3 = 0$ and in every row of A_5, there is atleast one non-zero element. ~~and every row of A_5 there is at least~~

Then, treat $r+1, r+2, \ldots, v$ are connected.

~~$(A_2) \neq 0$~~ $A_2 \neq 0$. If every row of A_2 contains one non-zero element treat $1, 2, \ldots, v$ are conected

If not, we proceed in a manner de
te above and show that the design is
connected.

Rules:- For determine whether a
design is connected or not.

1. If every element of C is non-
zero, then the design is connec

2. If C contains a row (or col)
then the design is connected.

3. Consider the last row of C and fi
the one element of this row, if at
least 1 element in any row abov
these elements is non-zero the
design is connected.

Consider the equ normal equ
pre multiplying both sides by

$$\begin{bmatrix} 0 & I_b & -N'R^{-1} \end{bmatrix}$$

we get $D\underset{\sim}{\beta} = P$

where $P = B - N'R^{-1}T$

This equations are the reduced normal equations for block effects.

It can be shown that all elementary contrasts among block effects are estimable iff $\text{Rank}(D) = b - 1$.

Since, $\text{Rank}(C) + b = \text{Rank}(D) + v$.

If all elementary contrast among treatments are estimable, then so are among block effect. The vector P is called the adjusted block total.

27.05.2023

Theorem: $\text{Cov}(Q, P) = 0$ iff $N = \frac{rk'}{n}$

$$Q = (D_1 - D_1 D_2' K^{-1} D_2)\underset{\sim}{y}$$

$$P = (D_2 - D_2 D_1' R^{-1} D_1)\underset{\sim}{y}$$

$$\text{Cov}(Q, P) = \sigma^2 (D_1 - D_1 D_2' K^{-1} D_2)(D_2 - D_2 D_1' R^{-1} D_1)'$$

$$= \sigma^2 (N - D_1 D_1' R^{-1} N - NK^{-1} D_2 D_2' + NK^{-1} D_2 D_1' R^{-1} N)$$

$= \sigma^2 \left(\cancel{N} - \cancel{N} - N + N k^{-1} N' R^{-1} N \right)$

$= \sigma^2 \left(N k^{-1} N' R^{-1} N - N \right)$

Now, if $N = \frac{rk'}{n}$, then $\sigma^{-2} Cov(Q, P) = n^{-2} \left(rk' k^{-1} k r' R^{-1} \right.$

$= n^{-2} \left(rk' k^{-1} k J_v' N \right) - N = n^{-2} \left(rk' k^{-1} k k' \right) - N$

$= n^{-2} \left(r k^{-1} J_v R k' \right) - N = n^{-2} \left(n r k' - N \right) = \frac{rk}{n} - N = 0.$

Conversely, let, $Cov(Q, P) = 0$

$\Rightarrow N k^{-1} N' R^{-1} N - N = 0$

$\Rightarrow (R - C) R^{-1} N - N = 0$

Let, $R^{-1} N = A$

$\therefore CA = 0$

also, $CJ = 0$

Let, $a_1, a_2, \ldots, a_b$ are the columns of A, then $Ca_i = 0 \; \forall i =$

Since, the design is connected, $rank(C) = v - 1$.

$\therefore a_i = \alpha_i J$ for all $i = 1, 2, \ldots, b$.

$\therefore A = J_v \alpha' = J_v (\alpha_1, \ldots, \alpha_b)$

Now, $N = r \alpha'$

$\Rightarrow J_v' N = J_v' r \alpha'$

$\Rightarrow k' = n \alpha'$

$\Rightarrow \alpha' = \frac{k'}{n}$

$\therefore N = RA = R J_v \frac{k'}{n} = \frac{rk'}{n}.$

~~Defn :-~~

Defn:- A connected design is called orthogonal iff $N = \frac{rk'}{n}$. Designs which are not orthogonal is called non-orthogonal. It is clear that, if atleast one element of N is zero, then the design is non-orthogonal. A design with atleast one zero element in N is called an ~~incomplete block design~~ incomplete block design.

Defn:- A block design is said to be Variance Balanced if it permits the estimation of all estimable randomised treatment contrast with same variance.

Theorem:- A connected block design is variance balanced iff all the non-zero eigen values of C matrix are equal.

Corollary:- A connected block design is Variance-Balanced iff all it's C matrix has all it's diagonal elements are equal and all it's off diagonal elements are equal

i.e. $C = (a-b) I + bJJ'$ for some a, b.

Defn:- A block design is said to be equireplicate if $r_i = r \ \forall i$, is said to be proper if $k_j = k \ \forall j$ and binary if $n_{ij} = 1$ or 0.

27.05.2011

Theorem: In an equireplicate, proper, binary, variance balanced design, the incidence matrix N satisfies :-

$NN' = (r-\lambda)I + \lambda JJ'$, where λ is a scalar such that $\lambda(v-1) = r(k-1)$

Proof: For an equireplicate, proper, block design.

$$C = rI - \frac{NN'}{k}$$

and if the design is binary, the trace of C is $vr - \frac{vr}{k}$

$= vr - b$.

Since, C is a doubly-centred matrix, we can show that using corrolary

$C = \theta(I - v^{-1}JJ')$

where θ is a unique non-zero eigen value of C.

$\therefore$ trace $(C) = \theta(v-1)$

$\therefore \theta = \frac{vr-b}{v-1}$

$$C = \frac{vr-b}{v-1}\left(I - v^{-1}JJ'\right)$$

$$rI - \frac{NN'}{k} = \frac{vr-b}{v-1}\left(I - v^{-1}JJ'\right)$$

$$\Rightarrow \frac{NN'}{k} = rI - \frac{vr-b}{v-1}\left(I - v^{-1}JJ'\right)$$

$$= \frac{b-r}{v-1}I + \frac{vr - \frac{vr}{k}}{v(v-1)}JJ'$$

$$= \frac{\frac{vr}{k} - r}{r(k-1)}\lambda I + \frac{r(k-1)}{k(v-1)}JJ'$$

$$= \frac{v-k}{k(k-1)}\lambda I + \frac{\lambda}{k} JJ'$$

$$\Rightarrow NN' = \frac{v\lambda - k\lambda}{k-1} I + \lambda JJ'$$

$$= \frac{rk + \lambda - r - k\lambda}{k-1} I + \lambda JJ'$$

$$= (r-\lambda) I + \lambda JJ'$$

Def:- Block design with incidence matrix sat the condition of above the theorem are called Balanced incomplete block design.

Theorem:- If the incidence matrix is not ha identical rows, then $b \geq v$ for all connected equireplicate variance balanced design.

proof:- For a connected var - balanced blo design, we have

$$C = \theta \left(I - v^{-1} JJ' \right)$$

Thus, the eigen values of C are 0 wi multiplicity unity and θ with multiplici $(v-1)$.

Hence, for an equireplicate design, the eigen values of $P = NK^{-1}N' = R - C$

$$= (r-\theta) I + \theta v^{-1} JJ$$

are r and $r-\theta$ with respective multiplicities 1 and $v-1$.

Thus, P is singular iff $r = \theta$.

In such a case, P and hence m is of rank unity, and the columns of P and hence those of MN are spanned by the vector J it follows then that in case $r = \theta$, the rows of N are all identical, which is a contradiction.

Hence, PQ is non-singular.

$\therefore \rho v = \rho(P) = r(MN) \leq 1$.

Ex:-

1. Let, $A = (a-b)I + bJJ'$, a, and b are scalar.

(i) the eigen values of A are $\theta_1 = a-b$. with multiplicity $n-1$ and $\theta_2 = a + (n-1)b$ with multiplicity unity.

(ii) $\det(A) = (a-b)^{n-1}\{a+(n-1)b\}$.

(iii) $A^{-1} = (c-d)I + dJJ'$

where $c = \frac{\{a+(n-2)b\}}{(a-b)\{a+(n-1)b\}}$, $d = \frac{-b}{(a-b)\{a+(n-1)b\}}$

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2. Show that, a necessary & sufficient co for a symmetric matrix A of order n to have all its diagonal elements equal and all its off-diagonal elements equal is that A has only 2 eigen values, one of multiplicity $(n-1)$ & the vector J is an eigen vector corresponding to the other eigen v

3. If Q denotes the vector of adjusted trea totals, show that

(i) $Q'J = 0$

(ii) $E(Q) = C\tau$

(iii) $D(Q) = \sigma^2 C$

4. Show that, for a connected design, th diagonal elements of C matrix are al positive.

5. Show that, for a connected design, $C + \frac{rr'}{n}$ is non-singular, hence show that, the var of the estimated Contrast $l'\tau$ is $\sigma^2 l' \left(C + \frac{rr'}{n}\right)^{-1} l$

For a connected equireplicate design, let

$$p-1 = k - \frac{N'N}{r} + \frac{kk'}{vr}$$

Show that, a solution of the normal equations

$$C\tau = Q \text{ is}$$

$$\hat{\tau} = \left(\frac{I}{r} + \frac{NPN'}{r^2}\right) Q$$

Moreover, for variance-balanced block design show that $\hat{\tau} = \theta^{-1} Q$.

7. Show that $C + a JJ'$ is non-singular, for a connected design.

8. Let the incidence matrix of a design with $v = 5$, $b = 8$, be as below

$$N = \begin{bmatrix} 1 & 1 & 1 & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Show that, the design is var-balanced.

9. Show that a n.s condition for a connected block design to be var-balaced is that if C matrix is of the form

$$C = (a-b)\,I + b\,JJ'$$

10. Prove that, a BIBD is always connected unless $k = 1$.

1. Test whether the following design is var. balanced or not

(1, 2); (1, 3); (1, 4); (2, 3); (2, 4); (3, 4)

(1, 2, 5, 5); (1, 3, 5, 5); (1, 4, 5, 5); (2, 3, 5, 5);

(2, 4, 5, 5); (3, 4, 5, 5).

Optimality of block design: Consider the intra-block model with usual paramet v, b, k, r, λ. Let, the inference problem

P: $\eta = L\tau$

with ~~$L\tau = 0$~~ ~~$bJ = 0$~~ $LJ = 0$.

where L is a $(p \times v)$ matrix of known elements $(p \geq 1)$. With reference to P a design, d is said to be acceptable if all components of η (which represents some contrasts among ~~some~~ 1 treatment effects) are

estimable under d.

Let $\mathcal{C}_p$ denotes the class of all acceptable design with reference to P. We say, the problem P is to be a non-singular full rank of $(L) = p = v - 1$.

Consequently, $\mathcal{C}_p$ consists of all connected design. We consider only non-singular full rank problem.

For any design, $d \in \mathcal{C}_p$. Let V_d denotes the dispersion matrix of $\hat{\eta}$ using d.

A optimality :- A design $d^* \in \mathcal{C}_p$ is said to be A optimality in $\mathcal{C}_p$ if

$$tr(V_{d^*}) \leq tr(V_d)$$

for any other design $d \in \mathcal{C}_p$

Defn :-

D- optimality :- A design $d^* \in \mathcal{C}_p$ is said to be D-optimal in $\mathcal{C}_p$ if

$$\det(V_{d^*}) \leq \det(V_d) \text{ for any } d \in \mathcal{C}_p$$

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E-optimality :- A design $d^* \in \Xi_p$ is said to be E-optimal in Ξ_p if $\max \lambda_{d^*} \leq \max \lambda_d$ where $\max \lambda_{d^*}$ (respectively: $\max \lambda_d$) is the largest eigen value of V_{d^*} (respectively: V_d) for any $d \in \Xi_p$

The A-optimality criteria chooses that design for which the average var of the estimate of η is the least. Similarly, the D-optimality criteria chooses the design for which generalised variance of the estimated parameters vector η is min. The E-optimality criteria relates to the minimisation of maximum var of estimates of all normalized treatment contrast.

Theorem :- For given, v, b, k $(< v)$, A-BIBD is A-optimal for estimating all elementary contrast in the class of all connected incomplete block design.

Proof:- Let V_d be the disp matrix of a set of $v-1$ LIM estimated elementary treatment contrast $\hat{\tau}$ using a design d, then it can be shown that

$$H(V_d) \propto \sum_{i=1}^{v-1} \theta_i^{-1} = \frac{(v-1)}{H}$$

where $\theta_1, \theta_2, \ldots, \theta_{v-1}$ are the non-zero eigen values of the C-matrix of d and H is the HM of the θ_i's.

From $HM \leq AM$ and we get

$$H \leq (v-1)^{-1} \sum_i \theta_i$$

$$= (v-1)^{-1} \operatorname{tr}(C)$$

Now, since $\operatorname{tr}(C) = \sum_{i=1}^{v} \left(r_i - k^{-1} \sum_{j=1}^{b} n_{ij}^2 \right)$

$$= n - k^{-1} \sum_{j=1}^{b} n_{ij}^2$$

It follows that,

$\operatorname{tr}(C)$ is minimized only for a binary design.

Now,

$$H = (v-1)^{-1} \cdot tr(C)$$

Iff all θ_i's are ~~bold~~ equal i.e. if the design is variance - balanced.

This implies that a BIBD if exists is A-optimal in the class of all connected block design.

Theorem :-

Ex:- 1. An equireplicate proper design is called a balanced ternary design if it's incidence matrix $N = (n_{ij})$ has the following properties

(i) $n_{ij} = 0, 1$ or 2 and

(ii) $\sum_j n_{ij} n_{mj} = \lambda$ for all $i \neq m$

$i, m = 1, 2, \ldots, v$.

Show that for such a design

$$\Delta = \sum_{j=1}^{b} n_{ij}^2 = rk - \lambda(v-1)$$

Suppose, D is a connected block design. Show that the variance of the BLUE of an elementary treatment contrast lies between $2\sigma^2\lambda^{-1}_{max}$ and $2\sigma^2\lambda^{-1}_{min}$,

λ^{-1}_{max} & λ^{-1}_{min} are max and least eigen values of C.

3. In a connected design, show that the avg. variance of all elementary treatment contrast is $\frac{2\sigma^2}{n}$, n is n.m. of non-zero eigen-values of C.

Theorem:- For a given b, v, k (< v) a BIBD (if exists) is E-optimal for infering on any full set of orthogonal contrast in the class of binary proper design.

Proof:- Let, d denotes a design with parameters b, v, k.

Let, m/τ' be an estimable treatment contrast.

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The BLUE of $m'T$ is

~~BLUE is~~ $m'\hat{\tau}$, where $\hat{\tau} = C^{-}Q$.

The variance of $m'\hat{\tau}$ is $\sigma^2 m' C^{-} m$ using design d.

If θ_{min} denotes the least positive eigen values of C, then we have

$$\max \frac{m' C^{-} m}{m'm} = \theta_{min}^{-1}$$

But for any connected design,

$$\theta_{min} \leq \bar{\theta} = (v-1)^{-1} \sum \theta_i = \frac{\lambda v}{k},$$

where $\lambda = \frac{bk(k-1)}{v(v-1)}$.

Now, if we are able to show

$$\theta_{min} = \frac{\lambda v}{k}.$$

Iff the design is a BIBD we have through.

If the design is a BIBD

$\theta_{min} = \frac{\lambda v}{k}$ as all the non-zero

eigen values of the matrix C of a BIBD are equal to $\frac{\lambda v}{k}$, as all the we are done.

Conversely let $\theta_{min} = \frac{\lambda v}{k}$, then all & others $(v-2)$ +ve eigen values must be $\frac{\lambda v}{k}$

Hence, $C = \left(\frac{\lambda v}{k}\right)\left(I - v^{-1} JJ'\right)$

and using the fact that, the design is binary & proper, this means the design is BIBD (If exists).

Defn:- A block design with b blocks, v treat, and block size k and incidence matrix $N = (n_{ij})$ is said to be Balanced Block design (BBD) if the following conditions are satisfied

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(i) $r_i = \sum_j n_{ij} = r \quad \forall\ i = 1, 2, \ldots, v$

(ii) $\lambda_{ih} = \sum_j n_{ij} n_{hj} = \lambda \quad \forall\ i \neq h, \quad i, h = 1, 2, \ldots, v$

(iii) $\left| n_{ij} - \frac{k}{v} \right| < 1$

If $k < v$, a BBD reduced to a BIB D.

Theorem :- For given b, v, k a BBD whenev. it exists is A-optimal, D-optimal, E-optimal, in C_p, for $P: \eta = LT$, where L is of the order $(v-1) \times v$ and the matrix $\begin{bmatrix} v^{-\frac{1}{2}} J' \\ L \end{bmatrix}$ is orthogonal.

i.e. for infering on any full set of orthonormal treatment contrast.

23.06.2011

~~Recovery of inter-block information:~~

When we discuss the intra-block analysis of block design the block and the treatment effects are fixed. Now, we analized the block designs with block effects are random.

~~Cons.~~

We consider binary and proper design.

Consider the model

$$y_{ij} = \mu + \tau_i + \beta_j + e_{ij}$$

β_j's are random variable such that

(i) $E(\beta_j) = 0$,

$Var(\beta_j) = \sigma_b^2$,

$Cov(\beta_j, \beta_{j'}) = 0 \quad \forall j \neq j' = 1, 2, \ldots, b$.

(ii) β_j's are uncorrelated with a errors e_{ij}

Now, the block total B_j's are given by

$$B_j = k\mu + \sum_i n_{ij}\tau_i + (k\beta_j + \sum_i e_{ij})$$

The new error terms denoted by

$$d_j = k\beta_j + \sum_i e_{ij}$$

Then, $E(d_j) = 0$,

$$Var(d_j) = k^2\sigma_b^2 + k\sigma^2$$

The intra-block estimate are then obtained by minimising the S.S. due to new errors i.e.

$$\sum_{j=1}^{b} d_j^2 = \sum_{j=1}^{b} \left(B_j - k\mu - \sum n_{ij}\tau_i\right)^2$$

$$= \left(B - k\mu J - N'\tau\right)'\left(B - k\mu J - N'\tau\right)$$

The normal equations are

$$-2kB'J + 2k^2\mu b + 2kJ'N'\tau = 0$$

$$\Rightarrow k\mu b + J'R\tau = G \quad \ldots (*)$$

and $kRJ\mu + NN'\tau = NB \quad \ldots (**)$

Combining two equations, we can write

$$\begin{bmatrix} bk & J'R \\ kRJ & NN' \end{bmatrix} \begin{bmatrix} \mu \\ \tau \end{bmatrix} = \begin{bmatrix} G \\ NB \end{bmatrix}$$

Premultiplying both side by the non singular matrix

$$\begin{bmatrix} I & 0 \\ -\frac{RJ}{b} & I \end{bmatrix}$$

, we obtain

$$\begin{bmatrix} bk & J'R \\ 0 & NN' - R \end{bmatrix}$$

$$\begin{bmatrix} bk & J'R \\ 0 & NN' - RJJ'R/b \end{bmatrix} \begin{bmatrix} \mu \\ \hline b \end{bmatrix} = \begin{bmatrix} G \\ NB - GRJ/b \end{bmatrix}$$

Assuming $J'R\tau = 0$ and NN' are non-singular

we get $(NN')\tau = NB - \frac{GRJ}{b}$

i.e. $\tau = (NN')^{-1}\left(NB - \frac{GRJ}{b}\right)$

$= (NN')^{-1} N \left(B - \frac{GJ}{b}\right)$

$= (NN')^{-1}\left(NB - \frac{G}{bk} NkJ\right)$

$= (NN')^{-1} NB - \frac{GJ}{bk}$.

and $bk\mu + J'R\tau = G$ i.e. $\mu = \frac{G}{bk}$.

we now have two estimators of τ. the intra-block estimator is given $\hat{\tau}$.

$= C^{-}Q$

and the intra-block estimator

Let, $\psi = p'\tau$ be a contrast of treatment effects, the intra-block estimator of ψ, ψ_1^* (say) is

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$\psi_1 = p' \hat{\tau} = p' C^- Q$

with variance, $\text{Var}(\psi_1) = \sigma^2 p' C^- p$.

The intra-block estimator of ψ, say ψ_2 is

$$\psi_2 = p' (NN')^{-1} N B \left(B - \frac{GJ}{b}\right)$$

$$= p' (NN')^{-1} NB - p' J \left(\frac{G}{bk}\right)$$

$$= p' (NN')^{-1} NB.$$

with variance, $\text{Var}(\psi_2) = \sigma_d^2 p' (NN')^{-1} p$.

where $\sigma_d^2 = k(k\sigma_b^2 + \sigma^2)$

Two estimators ψ_1 and ψ_2 are uncorrelated, because $\text{Cov}(Q_i, B_j) = 0 \; \forall i,j$.

Now, we want to combine these two estimators to obtain an estimator with smallest variance.

The combined estimator is obtained by taking a weighted avg. of ψ_1 & ψ_2 weights being the inverse of the variances of the two estimators. Thus, the combined estimator ψ^* is given by

$$\psi^* = \frac{(\theta_1 \psi_1 + \theta_2 \psi_2)}{\theta_1 + \theta_2}$$

$$\begin{bmatrix} bk & J'R \\ 0 & NN' - RJJ'R/b \end{bmatrix} \begin{bmatrix} \mu \\ b \end{bmatrix} = \begin{bmatrix} G \\ NB - \frac{GRJ}{b} \end{bmatrix}$$

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we get $(NN')\tau = NB - \frac{GRJ}{b}$

i.e. $\tau = (NN')^{-1}\left(NB - \frac{GRJ}{b}\right)$

$= (NN')^{-1} N \left(B - \frac{GJ}{b}\right)$

$= (NN')^{-1}\left(NB - \frac{G}{bk} NKJ\right)$

$= (NN')^{-1} NB - \frac{GJ}{bk}$.

and $bk\mu + J'R\tau = G$ i.e. $\mu = \frac{G}{bk}$.

we now have two estimators of τ. the intra-block estimator is given $\hat{\tau}$..

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$$\psi_2 = p' (NN')^{-1} N \left(B \cdot \frac{GJ}{b}\right)$$

$$= p' (NN')^{-1} NB - p' J \left(\frac{G}{bk}\right)$$

$$= p' (NN')^{-1} NB.$$

with variance, $\text{Var}(\psi_2) = \sigma_d^2 \, p' (NN')^{-1} p$

where $\sigma_d^2 = k(k\sigma_b^2 + \sigma^2)$

Two estimators ψ_1 and ψ_2 are uncorrelated, because $\text{Cov}(Q_i, B_j) = 0 \; \forall i, j$.

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$$\psi^* = \frac{(\theta_1 \psi_1 + \theta_2 \psi_2)}{\theta_1 + \theta_2}$$

where $\theta_1^{-1} = (p'c + p\sigma^2)$

$\theta_2^{-1} = p'(NN')^{-1} + p\sigma_d^2$

but σ^2 & σ_d^2 are usually unknown.

Alternative method due to Bose :-

As before, the model is

$$y = \mu J + D_1'\tau + D_2'\beta + e$$

$$= (J \; D_1') \begin{pmatrix} \mu \\ \tau \end{pmatrix} + D_2'\beta + e$$

With assumptions,

$$E(e) = 0, \quad D(e) = \sigma^2 I, \quad E(\beta) = 0$$

$$D(\beta) = \sigma_b^2 I, \quad Cov(e, \beta) = 0.$$

If we order the n obs's such that the 1st k comes from the 1st block, the next k comes from the 2nd block ... etc. Then, we have

$$D_2' D_2 = \text{diag}(J_{kk}, J_{kk}, \ldots, J_{kk})$$

Thus, $D(y) = \Sigma = \sigma^2 I + \sigma_b^2 D_2' D_2$

$$= \text{diag}(1, L, \ldots, L),$$

where $L = \sigma^2 I_k + \sigma_b^2 J_{kk}$.

Now, $\Sigma^{-1} = \text{diag}\,(L^{-1}, L^{-1}, \ldots, L^{-1})$

where, $L^{-1} = \alpha I_k + \beta J_{kk}$

$$\alpha = \sigma^{-2}, \quad \beta = -\frac{\sigma_b^2}{\sigma^2(\sigma^2 + k\sigma_b^2)}$$

Define, $w_1 = \sigma^{-2}$, $\quad w_2 = \dfrac{1}{\sigma^2 + k\sigma_b^2}$

then, $\alpha = w_1$, and $\beta = -\dfrac{(w_1 - w_2)}{k}$

So that, $\Sigma^{-1} = w_1 I_n - \left(\dfrac{w_1 - w_2}{k}\right) \text{diag}\,(J_{kk}, \ldots, J_{kk})$

$$= w_1 I_n - (w_1 - w_2)\frac{D_2' D_2}{k}$$

Now, the normal equation for estimating linear functions of $\mu, \tau_1, \tau_2, \ldots, \tau_v$ and

$$\begin{pmatrix} J' \\ D \end{pmatrix} \Sigma^{-1} \begin{pmatrix} J & D_1' \end{pmatrix} \begin{pmatrix} \mu \\ \tau \end{pmatrix} = \begin{pmatrix} J' \\ D_1 \end{pmatrix} \Sigma^{-1}$$

Now, $J'\Sigma^{-1} J = nw_1 - (w_1 - w_2)\dfrac{nk}{k}$

$$= nw_2$$

Similarly,

$$J'\Sigma^{-1} D_1' = w_2 r'$$

$D_1 \Sigma^{-1} D_1' = w_1 R - (w_1 - w_2)\frac{NN'}{k}$.

$J' \Sigma^{-1} y = w_1 G$ and $D_1 \Sigma^{-1} y = w_1 Q + w_2 (T - Q)$

∴ Normal equation reduce to

$$\begin{bmatrix} n w_1 & w_2 r' \\ w_2 r & w_1 R - (w_1 - w_2)\frac{NN'}{k} \end{bmatrix} \begin{bmatrix} \mu \\ \tau \end{bmatrix} = \begin{bmatrix} w_2 G \\ w_1 Q + w_2 (T - Q) \end{bmatrix}$$

Eliminating μ, we get

$$\left[w_1 \left(R - \frac{NN'}{k} \right) + w_2 \left(\frac{NN'}{k} - \frac{rr'}{n} \right) \right] \tau = w_1 Q + w_2 \left(T - Q - \frac{Gr}{n} \right)$$

Let, $C^* = \frac{NN'}{k} - \frac{rr'}{n}$

and $Q^* = T - Q - \frac{Gr}{n}$,

then $(w_1 C + w_2 C^*) \tau = w_1 Q + w_2 Q^*$

This is known as adjusted intra-block

normal equation.

Also, $E(Q) = C'L$ and $I(Q^*) = C^*C$

Now, $C^*J = 0$

Since, the best estimate of an estima function of treatment effects is of the form $q'(w_1 Q + w_2 Q^*)$.

It follows that if a linear fn of the treatment effect is estimable, then it must be a contrast. In practice, w_1 & w_2 will be known.