• Stochastic Processes

A stochastic process $\underline{X} = \{X(t), t \in T\}$ is a collection of random variables. That is, for each $t \in T$, $X(t)$ is a random variables.

We often interpret t as time and call $X(t)$ the state of the process at time t.

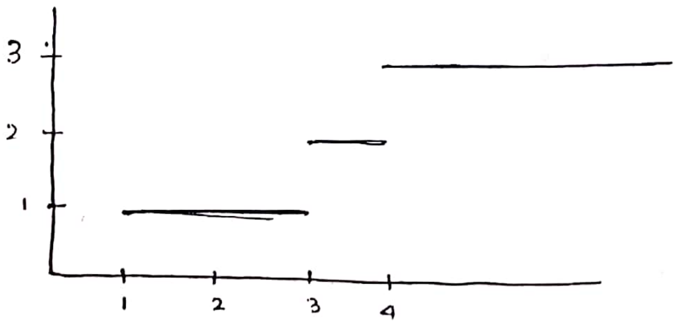
If the index set T is countable set, we call $\underline{X}$ a discrete-time stochastic process.

If T is a continuum, we call it a continuous time process.

• Sample path

Any ~~reta~~ realization of $\underline{X}$ is called a sample path.

As an example, if events are occuring randomly in time and $X(t)$ represents the number of events that occur in $[0,t]$ then a sample path of $\underline{X}$ may be as follows:



P.T.O

The above sample sample path of $\underline{X}$ corresponds to the initial event occurring at time 1, the next event at time 3 and the third at time 4 and no events anywhere else.

- ## Independent increments

A continuous-time stochastic process $\{X(t), t \in T\}$ is said to have independent increments if for all $t_0 < t_1 < t_2 < \dots < t_n$, the random variables

$$X(t_1) - X(t_0),\ X(t_2) - X(t_1), \dots, X(t_n) - X(t_{n-1})$$

are independent. i.e. it possesses independent increments if the changes in the processes' value over nonoverlapping time intervals are independent.

- ## Stationary increments

A continuous-time stochastic process $\{X(t), t \in T\}$ is said to have stationary increments if $X(t+s) - X(t)$ has the same distribution for all t.

i.e, it ~~pro~~ possesses stationary increments if the distribution of the change in value between any two points depends only on the distance between those point.

- ## Counting process

A stochastic process $\{N(t), t \geq 0\}$ is said to be a counting process if $N(t)$ represents the total number of 'events' that have occurred up to time t. Hence,

(i) $N(t) \geq 0$

(ii) $N(t)$ is integer valued random variable.

(iii) If $s < t$, then $N(s) \leq N(t)$.

(iv) For $s < t$, $N(t) - N(s)$ equals the number of events that have occured in the interval $(s, t]$

- ## ~~p~~ Poisson process

A counting process $\{N(t), t \geq 0\}$ is said to be a poisson process having rate λ, $\lambda > 0$, if:

(i) $N(0) = 0$

(ii) The process has independent increments.

(iii) The number of events in any interval of length t is Poisson distributed with mean λt.

That is, for all $s, t \geq 0$,

$$P\{N(t+s) - N(s) = n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!} \quad n = 0, 1, 2, \ldots$$

<u>Note</u> i) Poisson process has stationary increment

ii) $E[N(t)] = \lambda t$

Alternative Def^n

The counting process $\{N(t), t \geq 0\}$ is said to be a poisson process with rate λ, $\lambda > 0$, if

(i) $N(0) = 0$

(ii) The process has stationary and independent increments.

(iii) $P\{N(h) = 1\} = \lambda h + o(h)$

(iv) $P\{N(h) \geq 2\} = o(h)$

$\boxed{\text{P.T.O}}$

Theorem Two defn's are equivalent.

Pf Assume the first defn.

Then (i) and (ii) follows immediately

as ~~defn~~

Now, $P\{N(h)=1\} = P\{N(h+0)-N(0)=1\} = e^{-\lambda h}\cdot\lambda h$

$$e^{-\lambda h}\cdot\lambda h = \lambda h + \lambda h\left(e^{-\lambda h}-1\right) = \lambda h + o(h)$$

Since, $\lim\limits_{h\to 0} \dfrac{\lambda h\left(e^{-\lambda h}-1\right)}{h} = \lim\limits_{h\to 0} \lambda\left(e^{-\lambda h}-1\right) = 0$

Again, $P\{N(h)\geq 2\} = P\{N(h+0)-N(0)\geq 2\}$

$$= \sum_{n=2}^{\infty} e^{-\lambda h}\frac{(\lambda h)^n}{n!}$$

$$= 1 - e^{-\lambda h}(1+\lambda h)$$

$$= h\left(\frac{1-e^{-\lambda h}}{h} - e^{-\lambda h}\cdot\lambda\right)$$

Now, $\dfrac{1-e^{-\lambda h}}{h} = e^{-\lambda h}\cdot\lambda\left(\dfrac{e^{\lambda h}-1}{\lambda h}\right) \to \lambda$

as $h\to 0$

and $e^{-\lambda h}\lambda \to \lambda$ as $h\to 0$

$\therefore \dfrac{P\{N(h)\geq 2\}}{h} \to 0$ as $h\to 0$

Now, we show the converse i.e.

Alternating defn $\Rightarrow$ first defn

Let, $P_n(t) = P\{N(t) = n\}$

$$P_0(t+h) = P\{N(t+h) = 0\}$$
$$= P\{N(t) = 0,\ N(t+h) - N(t) = 0\}$$
$$= P\{N(t) = 0\}\ P\{N(t+h) - N(t) = 0\}$$
$$= P_0(t)\ P\{N(h) = 0\}$$
$$= P_0(t)\left[1 - \lambda h + o(h)\right]$$

Hence, $\dfrac{P_0(t+h) - P_0(t)}{h} = -\lambda P_0(t) + \dfrac{o(h)}{h}$

$\therefore\ P_0'(t) = -\lambda P_0(t)$

Solving the differential equation we got

$P_0(t) = Ke^{-\lambda t}$

Since $P_0(0) = P\{N(0) = 0\} = 1 \Rightarrow K = 1$

$\therefore\ P_0(t) = e^{-\lambda t}$

Similarly for $n \geq 1$

$$P_n(t+h) = P\{N(t+h) = n\}$$
$$= P\{N(t) = n,\ N(t+h) - N(t) = 0\}$$
$$+ P\{N(t) = n-1,\ N(t+h) - N(t) = 1\}$$
$$+ P\{N(t+h) = n,\ N(t+h) - N(t) \geq 2\}$$
$$= P_n(t)\ P_0(h) + P_{n-1}(t)\ P_1(h) + o(h)$$
$$= (1 - \lambda h)\ P_n(t) + \lambda h\ P_{n-1}(t) + o(h)$$

P.T.O

Thus, $$\frac{P_n(t+h)-P_n(t)}{h} = -\lambda P_n(t) + \lambda P_{n-1}(t) + \frac{o(h)}{h}$$

$$\therefore P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t)$$

Or, $$e^{\lambda t}\left[P_n'(t) + \lambda P_n(t)\right] = \lambda e^{\lambda t} P_{n-1}(t)$$

Hence $$\frac{d}{dt}\left(e^{\lambda t} P_n(t)\right) = \lambda e^{\lambda t} P_{n-1}(t)$$

Put $n=1$ we get, $$\frac{d}{dt}\left(e^{\lambda t} P_1(t)\right) = \lambda$$

So $$P_1(t) = (\lambda t + c)e^{-\lambda t}$$

Since $P_1(0) = 0$.

Therefore, $$P_1(t) = \lambda t\, e^{-\lambda t}$$

By induction we can easily show that,

$$P_n(t) = e^{-\lambda t}\frac{(\lambda t)^n}{n!} \qquad \square$$

- To show $N(t)$ has a Poisson distribution using Poisson approximation to the binomial distribution

Subdivide the interval $[0,t]$ into k equal parts

Note that the probability of having 2 or more events in any subinterval i.e.

$$P\{2 \text{ or more events in any subinterval}\}$$

$$\leq \sum_{i=1}^{k} P\{2 \text{ or more events in the } i\text{th subinterval}\}$$

$$= k\, o(t/k)$$ i.e. $t/k \to 0$

$$= t \cdot \frac{o(t/k)}{(t/k)} \longrightarrow 0 \quad \text{as } k \to \infty$$

Hence, $N(t)$ will (with a probability going to 1) just equal the number of subintervals in which an event occurs.

However, by stationary and independent increments this number will have a binomial distribution with parameter k and $p = \lambda t/k + o(t/k)$

Hence by the Poisson ~~distribution~~ approximation to the binomial we see by letting $k \to \infty$ that $N(t)$ will have a Poisson distribution with mean equal to

$$\lim_{k\to\infty} k\left[\lambda \frac{t}{k} + o(t/k)\right] = \lambda t$$

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• Interarrival and Waiting time distribution

Consider a Poisson process, and let X_1 denote the time of the first event. Further, for $n \geq 1$, let X_n denote the time between the $(n-1)$th and the nth event. The sequence is called the sequence of interarrival times.

Now, $P(X_1 > t) = P(N(t) = 0) = e^{-\lambda t}$

$\therefore X_1$ has an exponetial distribution with mean $1/\lambda$.

Proposition X_n, $n = 1, 2, \ldots$ are independent identically distributed exponential random variables having mean $1/\lambda$.

Proof. X_1 is expontentially distributed random variable.

Now, $P(X_2 > t \mid X_1 = s) = P(0 \text{ events in } (s, s+t] \mid X_1 = s)$

$= P(0 \text{ events in } (s, s+t])$

$= \cancel{e^{-\lambda t}} P(0 \text{ events in } (0, t])$

$= P(X_1 > t) = e^{-\lambda t}$

Therefore, X_2 is also an exponential random variable with mean $1/\lambda$ and independent of X_1.

By induction similarly we can prove X_n is expontential distribution.

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Waiting time Distributions

Consider a Poisson process, let the sequence $\{X_n, n \geq 1\}$ is the sequence of ~~increment~~ interarrival times.

Let $S_n = \sum_{i=1}^{n} X_i$

S_n is the arrival time of the nth event, which also called the waiting time until the nth event.

Then S_n has a gamma distribution with parameters n and λ. That is, its p.d.f is

$$f(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}, \quad t \geq 0$$

The above could also have been desired ~~by~~ as follows

Note that, $N(t) \geq n \Leftrightarrow S_n \leq t$.

Hence, $P\{S_n \leq t\} = P\{N(t) \geq n\} = \sum_{j=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!}$

$\therefore$ pdf of S_n is, $f(t) = \frac{d}{dt} P\{S_n \leq t\} = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$

Another way of obtaining the density of S_n is to use the independent increment assumption as follows:

$$\begin{aligned} P\{t \leq S_n < t+\Delta t\} &= P\{N(t) = n-1, \text{ 1 event in } (t, t+\Delta t)\} \\ &= P\{N(t) = n-1\} P\{\text{1 event in } (t, t+\Delta t)\} \\ &= P(N(t) = n-1)(\lambda \Delta t + o(\Delta t)) \\ &= \frac{e^{-\lambda t} \cdot (\lambda t)^{n-1}}{(n-1)!} \lambda \Delta t + o(\Delta t) \end{aligned}$$

$$f_{S_n}(t) = \lim_{\Delta t \to 0} \frac{P\{t < S_n < t+\Delta t\}}{\Delta t} = \lambda \frac{e^{-\lambda t} (\lambda t)^{n-1}}{(n-1)!}$$

$\boxed{\text{P.T.O}}$

- **Another way of defining a Poisson process**

Let a sequence $\{X_n, n \geq 1\}$ of independent identically distributed exponential random variables each having mean $1/\lambda$.

Now let us define a counting process by saying that the nth event of this process occurs at time S_n, where

$$S_n = X_1 + X_2 + \cdots + X_n$$

The resultant counting process $\{N(t), t \geq 0\}$ will be Poisson with rate λ.

- Another approach to proving alternative definition of Poisson processes implies the first definition

Let, $P_n(t) = P\{N(t) = n\}$

$\therefore \quad P_0(t+s) = P\{N(t+s) = 0\}$

$= P\{N(t+s) - N(s) = 0, N(s) = 0\}$

$= P\{N(t+s) - N(s) = 0\} P(N(s) = 0)$

$= P(N(t) = 0) . P(N(s) = 0)$

$= P_0(t) P_0(s)$

Let $\{X_n, n \geq 1\}$ is the sequence of interarrival times.

Let $f_n(t) = P(X_n > t)$

then $f_1(t) = P(X_1 > t) = P(N(t) = 0)$

$$f_1'(t) = \lim_{h \to 0} \frac{f_1(t+h) - f_1(t)}{h} = \lim_{h \to 0} \frac{P(X_1 > t+h) - P(X_1 > t)}{h}$$

$$= \lim_{h \to 0} \frac{P(N(t+h) = 0) - P(N(t) = 0)}{h}$$

$$= \lim_{h \to 0} \frac{P_0(t+h) - P_0(t)}{h}$$

$$= \lim_{h \to 0} P_0(t) \left(\frac{P_0(h) - 1}{h} \right)$$

$$= \lim_{h \to 0} P_0(t) \left(-\lambda + \frac{o(h)}{h} \right)$$

$$= -\lambda f_1(t)$$

Since, $f_1(0) = 1$, $f_1(t) = e^{-\lambda t}$.

P.T.O

$$P(X_2 > t \mid X_1 = s) = P(0 \text{ event in } (s, s+t] \mid X_1 = s)$$

$$= P\{0 \text{ event in } (s, s+t]\}$$ (by independent increamen

$$= e^{-\lambda t}$$ (by stationary increments)

Therefore, from the above we conclude that X_2 is also an exponential r.v with mean $\frac{1}{\lambda}$.

Repeating the same argument yields X_n, $n = 1, 2, -$ are independently identically distributed exponentia random variables having mean $1/\lambda$.

Now let, $S_n = \sum_{i=1}^{n} X_i \qquad n \geq 1$,

then S_n has a gamma distribution with paramete n and λ.

Hence p.d.f of S_n is

$$f(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \qquad t \geq 0$$

Now, $N(t) \geq n \iff S_n \leq t$

$$\therefore P(N(t) \geq n) = \int_0^t \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$\therefore P(N(t) = n) = P(N(t) \geq n) - P(N(t) \geq n+1)$$

$$= \int_0^t \left\{ \lambda e^{-\lambda t} \cdot \frac{(\lambda t)^{n-1}}{(n-1)!} - \lambda e^{-\lambda t} \frac{(\lambda t)^n}{n!} \right\} dt$$

$$= \left[\lambda e^{-\lambda} \frac{(\lambda t)^n}{n!} \cdot \frac{1}{\lambda} \right]_0^t = e^{-\lambda} \cdot \frac{(\lambda t)^n}{n!}$$

- Conditional distribution of the Arrival Times

For, $s \le t$,

$$P\{X_1 < s \mid N(t)=1\} = \frac{P\{X_1 < s,\ N(t)=1\}}{P\{N(t)=1\}}$$

$$= \frac{P\{1 \text{ event in } [0,s),\ 0 \text{ event in } [s,t)\}}{P\{N(t)=1\}}$$

$$= \frac{P\{1 \text{ event in } [0,s)\}\, P\{0 \text{ event in } [s,t)\}}{\lambda t e^{-\lambda t}}$$

$$= \frac{\lambda s e^{-\lambda s} \cdot e^{-\lambda(t-s)}}{\lambda t e^{-\lambda t}} = \frac{s}{t}$$

i.e. ~~Has~~ if exactly one event of a Poisson process has taken place by time t, then the time of the event ~~should~~ uniformly distributed over $[0,t]$.

Theorem Given that $N(t) = n$, show that the n arrival times $S_1, S_2, \ldots, S_n$ have the same distribution as the order statistics corresponding to n independent random variables uniformly distributed on the interval $(0, t)$.

Pf let $0 < t_1 < t_2 < \ldots < t_{n+1} = t$ and let h_i be small enough so that $t_i + h_i < t_{i+1}$, $i = 1, 2, \ldots n$

Now, $P\{t_i \le S_i \le t_i + h_i,\ i = 1, 2, \ldots n \mid N(t) = n\}$

$$= \frac{P\{\text{exactly 1 event in } [t_i, t_i + h_i],\ i = 1, 2, \ldots n \text{ and no events elsewhere in } [0,t]\}}{P(N(t) = n)}$$

$$= \frac{\lambda h_1 e^{-\lambda h_1} \ldots \lambda h_n e^{-\lambda h_n} e^{-\lambda(t - h_1 - h_2 - \ldots - h_n)}}{e^{-\lambda t} (\lambda t)^n / n!}$$

$$= \frac{n!}{t^n} h_1 \cdot h_2 \ldots h_n$$

Hence, the conditional density of $S_1, S_2, \ldots S_n$ given that $N(t) = n$ is

$$f(t_1, t_2, \ldots t_n) = \lim_{\substack{h_i \to 0 \\ \forall i}} \frac{P\{t_i \le S_i \le t_i + h_i,\ i = 1, \ldots n \mid N(t) = n\}}{h_1 h_2 \ldots h_n}$$

$$= \frac{n!}{t^n}$$

which is the joint p.d.f of the order statistics of n independent ~~uniformly distributed~~ random variables distributed over $(0, t)$.

- Markov Chains

Let $\mathcal{S} \subseteq \mathbb{Z}$, $\mathcal{S}$ is known as state space; $\mathbb{Z}$ = set of integers.

Let X_n, $n \geq 0$ are random variables. The stoc. process $\{X_n, n \geq 0\}$ is said to have Markov property if

$$P(X_{n+1} = x_{n+1} \mid X_0 = x_0, \ldots, X_n = x_n) = P(X_{n+1} = x_{n+1} \mid X_n = x_n)$$

$\forall\ n \in \mathbb{N} \cup \{0\}$ and $\forall\ x_0, x_1, \ldots, x_{n+1} \in \mathcal{S}$

The stoc. process having Markov property is know as Markov chain.

The conditional probabilities $P(X_{n+1} = y \mid X_n = x)$ are called the transition probabilies of the chain.

A Markov chains is said to have stationary transition probabilities if $P(X_{n+1} = y \mid X_n = x)$ is independent of n.

FROM NOW ON, when we say that $X_n, n \geq 0$ forms a Markov chain, we mean that these random variables satisfy the Markov property and have stationary transition probabilites.

- Markov chains having two states

Model: Consider a machine that at the start of any particular day is either broken down or in operating condition. Assume that if the machine is broken down at the start of the nth day, the probability is p that it will successfully repaired and in operating condition at the start of the $(n+1)$ th day. Assume also that if the machine is in operating condition at the start of the nth day, the probability is q that it will have a failure causing it to be broken down at the start of the $(n+1)$ th day.

Finally, let $\pi_0(0)$ denote the probability that the machine is broken down initially, i.e. at the start of the 0th day.

Let the state 0 correspond to the machine being broken down and let the state 1 correspond to the machine being in operating condition.

Let X_n be the random variable denoting the state of the machine at time n.

Hence, $P(X_{n+1} = 1 \mid X_n = 0) = p$

$P(X_{n+1} = 0 \mid X_n = 1) = q$

and $P(X_0 = 0) = \pi_0(0)$

P.T.O.

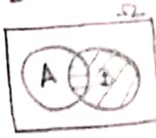
$[\because (A\cap B)\cup(A^c\cap B) = B$

$\therefore P(A\cap B)+P(A^c\cap B) = P(B)$

$\therefore P(A|B)+P(A^c|B) = 1$

$\therefore P(X_{n+1}=1|X_n=1)+P(X_{n+1}=0|X_n=1) = 1$

$\therefore P(X_{n+1}=1|X_n=1) = 1-q\,]$



$]$.

It follows immediately that

$$P(X_{n+1}=0 \mid X_n=0) = 1-p$$
$$P(X_{n+1}=1 \mid X_n=1) = 1-q$$

and $\pi_0(1) = P(X_0=1) = 1-\pi_0(0)$

$[\because A = (A\cap B)\cup(A\cap B^c) \therefore P(A) = P(A\cap B)+P(A\cap B^c)]$

Now, $\therefore P(X_{n+1}=0) = P(X_n=0, X_{n+1}=0) + P(X_n=1, X_{n+1}=0)$

$$= P(X_{n+1}=0 \mid X_n=0)\,P(X_n=0) + P(X_{n+1}=0 \mid X_n=1)\,P(X_n=1)$$

$$= (1-p)\,P(X_n=0) + q\,(1-P(X_n=0))$$

$$= (1-p-q)\,P(X_n=0) + q$$

$$\therefore P(X_1=0) = (1-p-q)\,\pi_0(0) + q$$

$$= (1-p-q)\,P(X_1=0) + q = (1-p-q)^2\pi_0(0) + q(1-p-q) + q$$

and $P(X_2=0) = (1-p-q)^2\,\pi_0(0) + q\,[1+(1-p-q)]$

Repeating n time we get,

$$P(X_n=0) = (1-p-q)^n\,\pi_0(0) + q\sum_{j=0}^{n-1}(1-p-q)^j$$

Assume that, $p+q>0$

then $\displaystyle\sum_{j=0}^{n-1}(1-p-q)^j = \frac{1-(1-p-q)^n}{p+q}$

$$\therefore P(X_n=0) = \frac{q}{p+q} + (1-p-q)^n\left(\pi_0(0) - \frac{q}{p+q}\right)$$

and $$P(X_n=1) = \frac{p}{p+q} + (1-p-q)^n\left(\pi_0(1) - \frac{p}{p+q}\right)$$

P.T.O

Suppose that p and q are neither both zero and nor both equal to 1.

Then $0 < p+q < 2$ i.e. $|1-p-q| < 1$ [$\because |r|<1$ in infinite G.P].

$\therefore \lim\limits_{n\to\infty} P(X_n=0) = \frac{q}{p+q}$ and $\lim\limits_{n\to\infty} P(X_n=1) = \frac{p}{p+q}$

• Joint distribution of $X_0, X_1, \ldots, X_n$

Let $n \geq 2$

$$P(X_0=x_0, X_1=x_1, X_2=x_2, \ldots, X_n=x_n)$$
$$= \pi_0(x_0)\, P(X_1=x_1 \mid X_0=x_0)\, P(X_2=x_2 \mid X_1=x_1) \ldots \ldots P(X_n=x_n \mid X_{n-1}=x_{n-1})$$

For example let $n=2$ then

x_0	x_1	x_2	$P(X_0=x_0, X_1=x_1 \text{ and } X_2=x_2)$
0	0	0	$\pi_0(0)(1-p)^2$
0	0	1	$\pi_0(0)(1-p)p$
0	1	0	$\pi_0(0)\,pq$
0	1	1	$\pi_0(0)p(1-q)$
1	0	0	$(1-\pi_0(0))\,q(1-p)$
1	0	1	$(1-\pi_0(0))\,qp$
1	1	0	$(1-\pi_0(0))(1-q)q$
1	1	1	$(1-\pi_0(0))(1-q)^2$

Transition function and initial distribution

Let X_n, $n \geq 0$ be a Markov Chain having state space $\mathcal{S}$.

The function $P(x, y)$, $x \in \mathcal{S}$ and $y \in \mathcal{S}$ defined by $P(x, y) = P(X_1 = y \mid X_0 = x)$ $x, y \in \mathcal{S}$ is called the transition function of the chain.

Now, $P(X_{n+1} = y \mid X_n = x) = P(x, y)$, $n \geq 1$

and $P(X_{n+1} = y \mid X_0 = x_0, \ldots, X_{n-1} = x_{n-1}, X_n = x) = P(x, y)$

$[\because P[X_{n+1} = y \mid X_0 = x_0, \ldots, X_{n-1} = x_{n-1}, X_n = x] = P[X_{n+1} = y \mid X_n = x]]$

$P(x, y)$ are called the one-step transition probabilities of the Markov chain.

The initial distribution of the chain is the function

$$\pi_0(x) = P(X_0 = x), \quad x \in \mathcal{S}$$

By induction we can show that,

$$P(X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n) = \pi_0(x_0) P(x_0, x_1) \cdots P(x_{n-1}, x_n)$$

We can define the transition function and initial distribution as follows:

We say $P(x, y)$, $x, y \in \mathcal{S}$, is a transition function if it satisfies $P(x, y) \geq 0 \quad \forall x, y \in \mathcal{S}$

and $\sum_y P(x, y) = 1 \quad \forall x \in \mathcal{S}$

and we say $\pi_0(x)$, $x \in \mathcal{S}$, is an initial distribution if it satisfies $\pi_0(x) \geq 0$, $x \in \mathcal{S}$

and $\sum_x \pi_0(x) = 1$

P.T.O.

It can be also shown that given any transition function P and any initial distribution π_0, there is a probability space and random variables $X_n, n \geq 0$ defined on that space satisfying

$$P(X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n) = \pi_0(x_0)\ P(x_0, x_1) \ldots P(x_{n-1}, x) \quad \text{— (1)}$$

<u>Exercise</u> show that these random variables form a Markov chain having transition function P and initial distribution π_0.

<u>Ans)</u>:→ Write definition of Markov Chain.

Then $P[X_{n+1} = x_{n+1} \mid X_0 = x_0, \ldots, X_n = x_n]$

$$= \frac{P[X_0 = x_0, \ldots, X_n = x_n, X_{n+1} = x_{n+1}]}{P[X_0 = x_0, \ldots, X_n = x_n]}$$

$$= \frac{\pi_0(x_0)\ P(x_0, x_1) \ldots P(x_{n-1}, x_n) P(x_n, x_{n+1})}{\pi_0(x_0)\ P(x_0, x_1) \ldots P(x_{n-1}, x_n)} \quad [\text{By } (1)]$$

$$= P(x_n, x_{n+1})$$

$$= P(X_{n+1} = x_{n+1} \mid X_n = x_n) \quad \forall\, n \in \mathbb{N} \cup \{0\},\ \forall\, x_0, \ldots, x_{n+1} \in S$$

So, these random variables form a Markov chain having transition function P and initial probabilities π_0.

- Random Walk

Let $\xi_1, \xi_2, \ldots$ be independent integer-valued random variables having common density f.

Let X_0 be be an integer-valued random variable that in independent of the ξ_i's and set $X_n = X_0 + \xi_1 + \xi_2 + \cdots + \xi_n$.

The sequence X_n, $n \geq 0$, is called a random walk.

It is a Markov chain whose state space is the integers and whose transition function is given by

$$P(x, y) = f(y-x)$$

To verify this, let π_0 denote the distribution of X_0.

Then, $P(X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n)$

$$= P(X_0 = x_0, \xi_1 = x_1 - x_0, \ldots, \xi_n = x_n - x_{n-1})$$

$$= P(X_0 = x_0)\, P(\xi_1 = x_1 - x_0) \cdots P(\xi_n = x_n - x_{n-1})$$

$$= \pi_0(x_0)\, f(x_1 - x_0) \cdots f(x_n - x_{n-1})$$

Now, $P(X_{n+1} = x_{n+1} \mid X_n = x_n, \ldots, X_0 = x_0)$

$$= \frac{P(X_{n+1} = x_{n+1}, X_n = x_n, \ldots, X_0 = x_0)}{P(X_n = x_n, \ldots, X_0 = x_0)}$$

$$= \frac{\pi_0(x_0)\, f(x_1 - x_0) \cdots f(x_n - x_{n-1})\, f(x_{n+1} - x_n)}{\pi_0(x_0)\, f(x_1 - x_0) \cdots f(x_n - x_{n-1})}$$

$$= f(x_{n+1} - x_n) = \cancel{P(X_{n+1} = x_{n+1} \mid X_n = x_n)}$$

And, $P(X_{n+1} = x_{n+1} \mid X_n = x_n)$

$$= \frac{P(X_n = x_n, \xi_{n+1} = x_{n+1} - x_n)}{P(X_n = x_n)}$$

$$= P(\xi_{n+1} = x_{n+1} - x_n) = f(x_{n+1} - x_n)$$

P.T.O.

Now, $P(x,y) = P(X_1 = y \mid X_0 = x)$
$= f(y-x)$

Also, $P(X_{n+1} = y \mid X_n = x) = f(y-x)$

Simple random walk

Let a particle undergo such a random walk. If the particle is in state x at a given observation, then by the next observation it will have jumped to state $x+1$ with probability p and to the state $x-1$ with probability q; and with probability r it will still be in state x.

Then the transition function is given by

$$P(x,y) = \begin{cases} p & y = x+1 \\ q & y = x-1 \\ r & y = x \\ 0 & \text{elsewhere} \end{cases}$$

and hence, $f(1) = p$; $f(-1) = q$ and $f(0) = r$

where, $p+q+r = 1$, $p,q,r > 0$

• Ehrenfest chain

It is a simple model of the exchange of heat or of gas molecules between two isolated bodies.

Suppose we have two boxes, labeled 1 and 2, and d balls labeled 1, 2, . . . d.

Initially some of these balls are in box 1 and the remainder are in box 2.

~~An integer is removed from its box and placed in the opposite~~ box.

An integer is selected at random from 1, 2, . . . , d, and the ball labeled by that integer is removed from its box and placed in the opposite box. The procedure is repeated infinitely with the selections being independent from ~~trail~~ trial to trial.

Let X_n denote the number of balls in box 1 after nth trial.

Then X_n, $n \geq 0$, is a Markov chain on $\mathcal{S} = \{0, 1, \ldots, d\}$

The transition function is

$$P(x, y) = \begin{cases} x/d & y = x-1 \\ 1 - x/d & y = x+1 \\ 0 & \text{elsewhere.} \end{cases}$$

• Birth and death chain

Consider a Markov chain either on $\mathcal{S} = \{0,1,2,\dots\}$ or on $\mathcal{S} = \{0,1,\dots,d\}$ such that starting from x the chain will be at $x-1$ or x or $x+1$ after one step. The transition function of such a chain is given by

$$P(x,y) = \begin{cases} q_x & y = x-1 \\ r_x & y = x \\ p_x & y = x+1 \\ 0 & \text{elsewhere} \end{cases}$$

where p_x, q_x and r_x are nonnegative ~~integer~~ numbers such that $p_x + q_x + r_x = 1$

A transition from state x to state $x+1$ corresponds to a "birth" and to state $x-1$ corresponds to a "death".

The Ehrenfest chain is an example of birth and death chain.

Another example is Gambler's ruin chain.

- Absorbing state

A state a of a Markov chain is called an absorving state if $P(a,a) = \cancel{\emptyset}\ 1$ or equivalently $P(a,y) = 0 \ \forall y \neq a$

Gambler's ruin chain

Suppose a grambler starts with a certain initial capital in dollars and makes a series of one dollar bets against the house.

Assume that he has respective probabilities p and $q = 1-p$ of winning and losing each bet, and that if his capital ever reaches zero, he is ruined and his capital remains zero thereafter.

Let X_n, $n \geq 0$ denote the gampler's capital at time n,

This is a Markov chain in which 0 is an absorbing state, and for $x \geq 1$

$$P(x,y) = \begin{cases} p & y = x+1 \\ q & y = x-1 \\ 0 & \text{elsewhere} \end{cases}$$

$, x \geq 1$

$P(0,0) = 1$

Such a chain is called a gambler's ruin chain on $\mathcal{S} = \{0, 1, 2, \ldots\}$

One can modify this model by supposing that if the capital of the gambler increase to d dollars he quits playing. In this case 0 and d are both absorbing states and $\mathcal{S} = \{0, 1, 2, \ldots, d\}$

$P(d,d) = 1$.

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• Queuing chain

Consider a service facility such as a checkout counter at a market. Those customers that have arrived at the facility but have not yet been served form a waiting line or queue. There are a variety of models to describe such system.

Let time be measured in minutes. Suppose that if there are any customers waiting for service at the beginning of any given period, exactly one customer will be served during the period, and that if ther are no customers waiting for service at the begining of a perid, none will be served during that period.

Let ξ_n denote the number of new customers arriving during the nth period. We assume that $\xi_1, \xi_2, \ldots$ are idependent nonnegative integer-valued random variables having common density f.

Let X_0 be the number of customers present initially, and for $n \geq 1$, let X_n denote the number of customers present at the end of the nth period.

If $X_n = 0$, then $X_{n+1} = \xi_{n+1}$ and if $X_n \geq 1$, then $X_{n+1} = X_n + \xi_{n+1} - 1$.

Then $X_n, n \geq 0$ is a Markov chain with state space as $\mathbb{N} \cup \{0\}$ and transition function P is given by

$$P(0, y) = f(y) \quad \text{and}$$

$$P(x, y) = f(y - x + 1), \quad x \geq 1.$$

- ## Branching chain

Consider particles such as ~~wate~~ neutrons or bacteria that can generate new particles of the same type. The initial set of objects is reffered to as belonging to the 0th generation. Particles generated from the nth generation are said to belong to the $(n+1)$th generation. Let $X_n, n \geq 0$, denote the number of particles in the nth generation.

Suppose that each particle gives rise ξ particles in the next generation, where ξ is a non-negative integer-valued random variable having density f. We suppose that the number of offspring of the various particle in the various generations are chosen independently according to the density f.

Under these assumptions $X_n, n \geq 0$, forms a Markov Chain whose state space is the nonnegative integers.

State 0 is an absorbing state.

For $x \geq 1$, $P(x,y) = P(\xi_1 + \xi_2 + \cdots + \xi_x = y)$

where $\xi_1, \xi_2, \ldots, \xi_x$ are independent random variables having common density f.

In particular, $P(1,y) = f(y)$, $y \geq 0$

Let $x \geq 1$,

$$P(X_{n+1} = y \mid X_0 = x_0, \ldots, X_{n-1} = x_{n-1}, X_n = x)$$

$$= \frac{P(X_0 = x_0, \ldots, X_{n-1} = x_{n-1}, X_n = x, X_{n+1} = y)}{P(X_0 = x_0, \ldots, X_{n-1} = x_{n-1}, X_n = x)}$$

P.T.O

Example (of Markov chain)

Consider a gene composed of d subunits, where d is some positive integer and each subunit is either normal or mutant in form.

Consider a cell with a gene composed of m mutant subunits and $d-m$ normal subunits. Before the cell divides into two daughter cells is composed of d units chosen at random from $2m$ mutant ~~sub~~ subunit and the $2(d-m)$ normal subunits. Suppose we follow a fixed line of descent from a given gene.

Let X_0 be the number of mutant subunits initially present and let X_n, $n \geq 1$, be the number present in the nth descendant gene.

Then X_n, $n \geq 0$, is a Markov chain on

$$\mathcal{S} = \{0, 1, 2, \ldots, d\} \text{ and}$$

$$P(x,y) = \frac{\binom{2x}{y}\binom{2d-2x}{d-y}}{\binom{2d}{d}}$$

~~Sta~~ with 0 and d as absorbing states.

H.T.

m-step transition function of the Markov chain

Let X_n, $n \geq 0$, be a Markov chain on $\mathcal{S}$ having transition function P.

Now, $P(X_{n+1} = x_{n+1}, \ldots, X_{n+m} = x_{n+m} \mid X_0 = x_0, \ldots, X_n = x_n)$

$$= P(x_n, x_{n+1})\, P(x_{n+1}, x_{n+2}) \cdots P(x_{n+m-1}, x_{n+m})$$

Let $A_0, A_1, \ldots, A_{n-1} \subseteq \mathcal{S}$ then

$$P(X_{n+1} = y_1, \ldots, X_{n+m} = y_m \mid X_0 \in A_0, \ldots, X_{n-1} \in A_{n-1}, X_n = x)$$

$$= P(x, y_1)\, P(y_1, y_2) \cdots P(y_{m-1}, y_m)$$

Let $B_1, B_2, \ldots B_m \subseteq \mathcal{S}$.

then $P(X_{n+1} \in B_1, \ldots, X_{n+m} \in B_m \mid X_0 \in A_0, \ldots, X_{n-1} \in A_{n-1}, X_n = x)$

$$= \sum_{y_1 \in B_1} \cdots \sum_{y_m \in B_m} P(x, y_1)\, P(y_1, y_2) \cdots P(y_{m-1}, y_m)$$

The m-step transition function $P^m(x, y)$ is defined by

$$P^m(x,y) = \sum_{y_1} \cdots \sum_{y_{m-1}} P(x, y_1)\, P(y_1, y_2) \cdots P(y_{m-1}, y)$$

$$= P(X_{n+m} = y \mid X_n = x)$$

$$= P(X_{n+m} = y \mid X_0 \in A_0, \ldots, X_{n-1} \in A_{n-1}, X_n = x)$$

for $m \geq 2$

and $P^1(x,y) = P(x,y)$

and $P^0(x,y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$

P.T.O

Again,

$$\hat{P}^{n+m}(x,y) = P(X_{n+m}=y \mid X_0=x)$$
$$= \sum_{z} P(X_n=z \mid X_0=x)\, P(X_{n+m}=y \mid X_0=x, X_n=z)$$
$$= \sum_{z} P^n(x,z)\, P(X_{n+m}=y \mid X_n=z)$$
$$= \sum_{z} P^n(x,z)\, P^m(z,y)$$

Let π_0 be an initial distribution for Markov chain.

Since, $P(X_n=y) = \sum_{x} P(X_0=x, X_n=y)$

$$= \sum_{x} P(X_0=x)\, P(X_n=y \mid X_0=x)$$
$$= \sum_{x} \pi_0(x)\, P^n(x,y)$$

Also, $P(X_{n+1}=y) = \sum_{x} P(X_n=x)\, P(x,y)$

If we know the distribution of X_0, we can find the distribution of X_1. Knowing the distribution of X_1 we can find the distribution of X_2.
Similarly, we can find the distribution of X_n.

We will use the notation $P_x(\ )$ to denote probabilities of events defined in terms of a Markov chain starting at x.
Thus, $P_x(X_1 \in B_1, X_2 \in B_2, \ldots, X_m \in B_m)$

$$= P(X_1 \in B_1, \ldots, X_m \in B_m \mid X_0 = x)$$

- **Hitting times**

Let $A \subseteq \mathcal{S}$, the hitting time T_A of A is defined by $T_A = \min\{n > 0 : X_n \in A\}$ if $X_n \in A$ for some $n > 0$ and by $T_A = \infty$ if $X_n \notin A$, $\forall\, n > 0$

For $a \in \mathcal{S}$, we denote $T_{\{a\}}$ by T_a.

Theorem $P^n(x,y) = \sum_{m=1}^{n} P_x(T_y = m)\, P^{n-m}(y,y)$, $n \geq 1$

Pf Note that the events $\{T_y = m, X_n = y\}$ $1 \leq m \leq n$ are dijoint and

$$(X_n = y) = \bigcup_{m=1}^{n} \{T_y = m, X_n = y)$$

$$\begin{aligned}
\therefore \quad P^n(x,y) &= P_x(X_n = y) \\
&= \sum_{m=1}^{n} P_x(T_y = m, X_n = y) \\
&= \sum_{m=1}^{n} P_x(T_y = m)\, P(X_n = y \mid X_0 = x, T_y = m) \\
&= \sum_{m=1}^{n} P_x(T_y = m)\, P(X_n = y \mid X_0 = x, X_1 \neq y, \ldots, X_{m-1} \neq y, X_m = y) \\
&= \sum_{m=1}^{n} P_x(T_y, m)\, P^{n-m}(y,y)
\end{aligned}$$

Theorem

If a is an absorbing state of a Markov chain X_n, $n \geq 0$, then $P^n(x,a) = P_x(T_a \leq n)$, $n \geq 1$

Pf If a is an absorbing state, then $P^{n-m}(a,a) = 1$ for $1 \leq m \leq n$ and hence

$$P^n(x,a) = \sum_{m=1}^{n} P_x(T_a = m)\, P^{n-m}(a,a)$$

$$= \sum_{m=1}^{n} P_x(T_a = m) = P_x(T_a \leq n)$$

Note:

1) $P_x(T_y = 1) = P_x(X_1 = y) = P(x,y)$

2) $P_x(T_y = 2) = \sum_{z \neq y} P(x,z)\, P(z,y)$

3) $P_x(T_y = n+1) = \sum_{z \neq y} P(x,z)\, P_z(T_y = n)$, $n \geq 1$

- **Transition matrix**

Let X_n, $n \geq 1$, be a Markov chain with finite state space $\mathcal{S} = \{0, 1, 2, \ldots, d\}$.

The transition matrix P is a $(d+1) \times (d+1)$ matrix with (x, y) th entry $P(x, y)$.

Similarly, P^n is n-step transition matrix with (x, y) th entry $P^n(x, y)$.

Clearly, $P^2(x, y) = \sum_z P(x, z) P(z, y)$

Hence, $P^2 = PP$

More generally, $P^{n+1} = P^n P$

It follows from above that the n-step transition matrix P^n is the n-th power of P.

An initial distribution π_0 can be thought of as a $(d+1)$-dimensional row vector $\pi_0 = (\pi_0(0), \ldots, \pi_0(d))$

If we let π_n denoted the $(d+1)$-dimentional row vector $\pi_n = (P(X_n = 0), \ldots, P(X_n = d))$

then, $\pi_n = \pi_0 P^n$.

and $\pi_{n+1} = \pi_n P$.

$\boxed{\text{P.T.O}}$

Example

The transition matrix of the gambler's ruin chain on $\{0,1,2,3\}$ is

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Example Consider the two-state Markov chain having one-step transition matrix

$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$$

where, $p+q>0$.

Then from previous discussion

$$P^n(0,0) = \frac{q}{p+q} + (1-p-q)^n \frac{p}{p+q} \quad (\text{set } \pi_0(0)=1)$$

$$P^n(0,1) = \frac{p}{p+q} - (1-p-q)^n \frac{p}{p+q} \quad (\text{set } \pi_0(1)=0)$$

$$P^n(1,0) = \frac{q}{p+q} - (1-p-q)^n \frac{q}{p+q} \quad (\text{set } \pi_0(0)=0)$$

$$P^n(1,1) = \frac{p}{p+q} + (1-p-q)^n \frac{q}{p+q} \quad (\text{set } \pi_0(1)=1)$$

Hence $$P^n = \frac{1}{p+q}\begin{bmatrix} q & p \\ q & p \end{bmatrix} + \frac{(1-p-q)^n}{p+q}\begin{bmatrix} p & -p \\ -q & q \end{bmatrix}$$

Transient and recurrent states

Let $X_n, n \geq 0$, be a Markov chain having state space $\mathcal{S}$ and transition function P.

Set $\rho_{xy} = P_x(T_y < \infty)$

A state y is called recurrent if $\rho_{yy} = 1$ and transient if $\rho_{yy} < 1$.

Clearly an absorbing state is ~~reco~~ recurrent.

Let $1_y(z) = \begin{cases} 1 & z = y \\ 0 & z \neq y \end{cases}$

Let $N(y)$ denote the number of times $n \geq 1$ that the chain is in state y.

Then, $N(y) = \sum_{n=1}^{\infty} 1_y(X_n)$

and $\{N(y) \geq 1\} = \{T_y < \infty\}$

Thus $P_x(N(y) \geq 1) = P_x(T_y < \infty) = \rho_{xy}$

and
$$P_x(N(y) \geq 2) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} P_x(T_y = m)\, P_y(T_y = n)$$
$$= \left(\sum_{m=1}^{\infty} P_x(T_y = m)\right)\left(\sum_{n=1}^{\infty} P_y(T_y = n)\right)$$
$$= \rho_{xy}\rho_{yy}$$

Similarly $P_x(N(y) \geq m) = \rho_{xy}\rho_{yy}^{m-1} \qquad m \geq 1$

Now, ~~Since~~ $P_x(N(y) = m) = \rho_{xy}\rho_{yy}^{m} - \rho_{xy}\rho_{yy}^{m-1}$

$$= \rho_{xy}\rho_{yy}^{m-1}(1 - \rho_{xy}), \quad m \geq 1$$

and $P_x(N(y) = 0) = 1 - \rho_{xy}$.

P.T.O

Let us use the notation $E_x(\cdot)$ to denote expectations of random variables defined in terms of a Markov chain starting at x.

Then, $E_x(1_y(X_n)) = P_x(X_n = y) = P^n(x,y)$

and $E_x(N(y)) = E_x\left(\sum_{n=1}^{\infty} 1_y(X_n)\right)$

$$= \sum_{n=1}^{\infty} P^n(x,y)$$

Hence, $G(x,y) = E_x(N(y))$ ~~is~~ denotes the expected number of visits to y for a Markov chain starting at x.

Theorem (i) Let y be a transient state. Then

$$P_x(N(y) < \infty) = 1 \quad \text{and} \quad G(x,y) = \frac{\rho_{xy}}{1-\rho_{yy}}, \quad x \in \mathcal{S}$$

(ii) Let y be a recurrent state.

Then $P_y(N(y) = \infty) = 1$ and $G(y,y) = \infty$

Also, $P_x(N(y) = \infty) = P_x(T_y < \infty) = \rho_{xy}$, $x \in \mathcal{S}$

If $\rho_{xy} = 0$, then $G(x,y) = 0$

while if $\rho_{xy} > 0$, then $G(x,y) = \infty$

Proof. Let y be a transient state.

Since, $0 \le \rho_{yy} < 1$, it follows that

$$P_x(N(y) = \infty) = \lim_{m \to \infty} P_x(N(y) \geq m) = \lim_{m\to\infty} \rho_{xy}\rho_{yy}^{m-1}$$
$$= 0$$

and
$$G(x,y) = E_x(N(y))$$
$$= \sum_{m=1}^{\infty} m\, P_x(N(y) = m)$$
$$= \sum_{m=1}^{\infty} m\, \rho_{xy}\rho_{yy}^{m-1}(1-\rho_{yy})$$
$$= \rho_{xy}(1-\rho_{yy}) \sum_{m=1}^{\infty} m\, \rho_{yy}^{m-1}$$
$$= \rho_{xy}(1-\rho_{yy}) \frac{1}{(1-\rho_{yy})^2} = \frac{\rho_{xy}}{1-\rho_{yy}} < \infty$$

Now let y be recurrent.

Then $\rho_{yy} = 1$ and, hence

$$P_x(N(y) = \infty) = \lim_{m\to\infty} P_x(N(y) \geq m) = \rho_{xy}$$

In particular, $P_y(N(y) = \infty) = 1$

$\therefore G(y,y) = E_y(N(y)) = \infty$ $\boxed{\text{P.T.O}}$

If $\rho_{xy} = 0$, then $P_x(T_y = m) = 0 \quad \forall\, m > 0$

So, $P^n(x,y) = \sum_{m=1}^{n} P_x(T_y = m)\, P^{n-m}(y,y) = 0 \quad , n \geq 1$

Thus, $G(x,y) = E_x(N(y)) = \sum_{n=1}^{\infty} P^n(x,y) = 0$

If $\rho_{xy} > 0$ then.

$$P_x(N(y) = \infty) = \rho_{xy} > 0$$

and hence, $G(x,y) = E_x(N(y)) = \infty$.

Note Let y be a transient state.

Since, $\sum_{n=1}^{\infty} P^n(x,y) = G(x,y) < \infty \quad , x \in \mathcal{S}$

we see that. $\lim_{n \to \infty} P^n(x,y) = 0 \quad , \quad \forall\, x \in \mathcal{S}$

Transient chain and recurrent chain

A Markov chain is called a transient chain if all of its states are transient and a recurrent chain if all its states are recurrent.

Theorem A Markov chain having a finite state space must have at least one recurrent state and hence cannot possibly be a transient chain.

Pf If possible let $\mathscr{S}$ is finite and all the states are transient,

Now, $\lim_{n\to\infty} P^n(x,y) = 0 \quad \forall\, y \in \mathscr{S}$

$$\therefore \lim_{n\to\infty} \sum_{y\in\mathscr{S}} P^n(x,y) = \sum_{y\in\mathscr{S}} \lim_{n\to\infty} P^n(x,y) = 0$$

But, $\sum_{y\in\mathscr{S}} P^n(x,y) = P_x(X_n \in \mathscr{S}) = 1$

A contradiction.

Problem: Let y be a transient state. Show that for all x:

$$\sum_{n=0}^{\infty} P^n(x,y) \le \sum_{n=0}^{\infty} P^n(y,y)$$

Soln

$$\sum_{n=0}^{\infty} P^n(x,y) = \sum_{n=0}^{\infty} P^n(y,y) \quad \text{if } x=y$$

Let $x \neq y$, then $P^0(x,y)=0$

$$\therefore \sum_{n=0}^{\infty} P^n(x,y) = \sum_{n=1}^{\infty} P^n(x,y)$$

$$= G(x,y)$$

$$= \frac{\rho_{xy}}{1-\rho_{yy}} \le \frac{1}{1-\rho_{yy}}$$

$$= 1+\frac{\rho_{yy}}{1-\rho_{yy}}$$

$$= 1+G(y,y)$$

$$= P^0(y,y)+\sum_{n=1}^{\infty} P^n(y,y)$$

$$= \sum_{n=0}^{\infty} P^n(y,y).$$

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* Definition

Let x and y be two not necessarily distinct states.
We say ~~x~~ that x leads to y if $\rho_{xy} > 0$.

Theorem 1) x leads to y iff $P^n(x,y) > 0$ for some positive integer n.

2) if x leads to y and y leads to z, then x leads to z.

Pf 1) Let $\rho_{xy} > 0$ then $G(x,y)$ is $\dfrac{\rho_{xy}}{1-\rho_{yy}} > 0$

or ∞ according as y be a transient state

or recurrent state.

Since $G(x,y) = \sum_{n=1}^{\infty} P^n(x,y)$

$\therefore$ $P^n(x,y) > 0$ for some n.

Conversely, let $P^n(x,y) > 0$ for some n

then $G(x,y) \neq 0$

if y be a transient state then

$G(x,y) = \dfrac{\rho_{xy}}{1-\rho_{yy}} \neq 0 \Rightarrow \rho_{xy} > 0$

and if y be a recurrent state then

$\rho_{xy} > 0$ otherwise if $\rho_{xy} = 0$ then $G(x,y) = 0$

P.T.O

2) Let $\rho_{xy}, \rho_{yz} > 0$

Then $P^n(x,y), P^m(y,z) > 0$ for some positive integer n and m.

Now, $\rho_{xz} = P_x(T_z < \infty)$

$$= P_x(N(z) \geq 1)$$

$$\geq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P_x(T_y = m)\, P_y(T_z = n)$$

$$= \rho_{xy}\, \rho_{yz} > 0$$

Hence $P^{n+m}(x,z) \geq P^n(x,y)\, P^m(y,z) > 0 \Rightarrow \rho_{xz} > 0$

<u>Theorem</u> Let x be a ~~recurrence~~ recurrent state and suppose that x leads to y. Then y is recurrent and $\rho_{xy} = \rho_{yx} = 1$.

<u>Pf</u> Omitted.

Problem Let $X_n, n \geq 0$, be a Markov chain. Show that

$$P(X_0 = x_0 \mid X_1 = x_1, \ldots, X_n = x_n) = P(X_0 = x_0 \mid X_1 = x_1)$$

Soln $P(X_0 = x_0 \mid X_1 = x_1, \ldots, X_n = x_n)$

$$= \frac{P(X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n)}{P(X_1 = x_1, \ldots, X_n = x_n)}$$

$$= \frac{P(X_0 = x_0)\, P(X_1 = x_1 \mid X_0 = x_0) \cdots P(X_n = x_n \mid X_{n-1} = x_{n-1}, \ldots, X_0 = x_0)}{P(X_1 = x_1)\, P(X_2 = x_2 \mid X_1 = x_1) \cdots P(X_n = x_n \mid X_{n-1} = x_{n-1}, \ldots, X_0 = x_0)}$$

$$= \frac{P(X_0 = x_0)\, P(X_1 = x_1 \mid X_0 = x_0) \cdots P(X_n = x_n \mid X_{n-1} = x_{n-1})}{P(X_1 = x_1)\, P(X_2 = x_2 \mid X_1 = x_1) \cdots P(X_n = x_n \mid X_{n-1} = x_{n-1})}$$

$$= \frac{P(X_0 = x_0)\, P(X_1 = x_1 \mid X_0 = x_0)}{P(X_1 = x_1)}$$

$$= \frac{P(X_0 = x_0, X_1 = x_1)}{P(X_1 = x_1)} = P(X_0 = x_0 \mid X_1 = x_1)$$

Problem Let $X_n, n \geq 0$ be a Markov chain. Show that

$$P(X_{n+1} = x_{n+1} \mid X_0 \in A_0, \ldots, X_{n-1} \in A_{n-1}, X_n = x_n) \cancel{= P(X_n = x_n \mid X_0}$$

$$= P(X_{n+1} = x_{n+1} \mid X_n = x_n)$$

and $P(X_0 = x_0 \mid X_1 = x_1, X_2 \in A_2, \ldots, X_n \in A_n)$

$$= P(X_0 = x_0 \mid X_1 = x_1)$$

Hint: Use the fact that if D_i are disjoint and $P(C \mid D_i) = p$ then $P(C \mid \cup_i D_i) = p$

$\boxed{\text{P.T.O}}$

Problem: Let $X_n, n \geq 0$ be a Markov Chain

Show that, $P(X_n = x_n \mid X_0 = x_0, \ldots, X_{n-1} = x_{n-1}, X_{n+1} = x_{n+1}, \ldots X_{n+m} = x_{n+m})$

$$= P(X_n = x_n \mid \cancel{X_0 = x_0}, X_{n-1} = x_{n-1}, X_{n+1} = x_{n+1})$$

Pf

$$P(X_n = x_n \mid X_0 = x_0, \ldots, X_{n-1} = x_{n-1}, X_{n+1} = x_{n+1}, \ldots, X_{n+m} = x_{n+m})$$

$$= \frac{P(X_0 = x_0, X_1 = x_1, \ldots, X_{n+m} = x_{n+m})}{P(X_0 = x_0, \ldots, X_{n-1} = x_{n-1}, X_{n+1} = x_{n+1}, \ldots, X_{n+m} = x_{n+m})}$$

$$= \frac{\cancel{P(X_{n+m} = x_{n+m} \mid X_0 = x_0, \ldots}, X_{n+m-1} = x_{n+m-1}) \times P(X_0 = x_0, \ldots, X_{n+m-1} = x_{n+m-1})}{\cancel{P(X_{n+m} = x_{n+m} \mid X_0 = x_0}, \ldots, X_{n-1} = x_{n-1}, X_{n+1} = x_{n+1}, \ldots, X_{n+m-1} = x_{n+m-1}) \times P(X_0 = x_0, \ldots, X_{n-1} = x_{n-1}, X_{n+1} = x_{n+1}, \ldots X_{n+m-1} = x_{n+m-1})}$$

$\cancel{P(X_{n+m} = x_{n+m} \mid X_{n+}}$

$$= \frac{P(X_0 = x_0 \mid X_1 = x_1, \ldots, X_{n+m-1} = x_{n+m-1}) P(X_1 = x_1, \ldots, X_{n+m-1} = x_{n+m-1})}{P(X_0 = x_0 \mid X_1 = x_1, \ldots, X_{n-1} = x_{n-1}, X_{n+1} = x_{n+1}, \ldots, X_{n+m-1} = x_{n+m-1}) \times P(X_1 = x_1, \ldots, X_{n-1} = x_{n-1}, X_{n+1} = x_{n+1}, \ldots, X_{n+m-1} = x_{n+m-1})}$$

$$= \frac{P(X_1 = x_1, \ldots, X_{n+m-1} = x_{n+m-1})}{P(X_1 = x_1, \ldots, X_{n-1} = x_{n-1}, X_{n+1} = x_{n+1}, \ldots, X_{n+m-1} = x_{n+m-1})}$$

Repeating these process we can show the ~~results~~ result.

Time-Reversible Markov chains

Consider now a stationary Markov chain having transition probabilities P_{ij} and stationary probabilities π_i, and suppose that starting at some time we trace the sequence of states going backwards in time. That is, starting at time n consider the sequence of states $X_n, X_{n-1}, \cdots$. It ~~turn~~ turns out that this sequence of states is itself a Markov chain with transition probabilities P_{ij}^* defined by

$$P_{ij}^* = P\{X_m = j \mid X_{m+1} = i\}$$

$$= \frac{P\{X_{m+1} = i \mid X_m = j\} P\{X_m = j\}}{P\{X_{m+1} = i\}}$$

$$= \frac{\pi_j P_{ji}}{\pi_i}$$

To show that the reversed process is indeed a Markov chain we need to verify that

$$P\{X_m = j \mid X_{m+1} = i, X_{m+2}, X_{m+3} \cdots\} = P(X_m = j \mid X_{m+1} = i)$$

To see that the above is true, think of the present time as being time $m+1$. Then since $X_n, n \geq 1$ is a Markov chain it follows that given the present state X_{m+1} the past state X_m

PTO

and the future state $X_{m+2}, X_{m+3}, \cdots$ are independent. Thus the reversed process is also a Markov Chain with transition probabilities given by

$$P_{ij}^* = \frac{\pi_j P_{ji}}{\pi_i}$$

If $P_{ij}^* = P_{ij}$ for all i, j, then the Markov Chain is said to be time reversible.

The condition for time reversibility, that

$$\pi_i P_{ij} = \pi_j P_{ji} \quad \text{for all } i, j.$$

which can be interpreted as stating that, for all states i and j, the rate at which the process goes from i to j (namely, $\pi_i P_{ij}$) is equal to the rate at which it goes from j to i (namply, $\pi_j P_{ji}$).

It should be noted that this is an obvious necessary condition for time reversibility since a transition from i to j going backward in time is equivalent to a transition from j to i going forward in time; that is, if $X_m = i$ and $X_{m-1} = j$, then a transition from i to j is observed if we are looking backward in time and one from j to i if we are looking forward in time.

• Martingales

A sequence of random variables $X_n, n \geq 0$ having the property

$$E\left[X_{n+1} \mid X_0 = x_0, X_1 = x_1, \ldots, X_{n-1} = x_{n-1}, X_n = x\right] = x$$

is called a martingales.

Consider a Markov chain having state space $\{0, 1, \ldots, d\}$ and transition function P such that

$$\sum_{y=0}^{d} y\, P(x, y) = x \quad , \quad x = 0, \ldots, d.$$

then $E\left[X_{n+1} \mid X_0 = x_0, \ldots, X_{n-1} = x_{n-1}, X_n = x\right]$

$$= \sum_{y=0}^{d} y\, P\left[X_{n+1} = y \mid X_0 = x_0, \ldots, X_{n-1} = x_{n-1}, X_n = x\right]$$

$$= \sum_{y=0}^{d} y\, P(x, y)$$

$$= x$$

Then the Markov chain is a Martingales.

If, $X_n, n \geq 0$ is ~~a~~ Markov chain which also a Martingale with state space $\{0, 1, \ldots, d\}$ then $\sum_{y=0}^{d} y\, P(0, y) = 0$

$\cancel{\sum_{y=0}^{d} y\, P[X_{n+1} = y \mid X_0 = x_0, \ldots, X_{n-1} = x_{n-1}}, X_n = 0] = 0$

which implies that 0 is an absorbing state.

Similarly, d is an absorbing state.

- Stationary Distributions of a Markov chain

Let $X_n, n \geq 0$ be a Markov chain having state space $\mathcal{S}$ and transition function P.

If $\pi(x), x \in \mathcal{S}$, are nonnegative numbers such that

$$\sum_{x \in \mathcal{S}} \pi(x) = 1 \quad \text{and}$$

$$\sum_{x \in \mathcal{S}} \pi(x) P(x,y) = \pi(y) \quad \forall\, y \in \mathcal{S}$$

then π is called a stationary distribution.

Suppose that a stationary distribution π exist and that $\lim_{n \to \infty} P^n(x,y) = \pi(y) \quad y \in \mathcal{S}$

then π is called the steady state distribution.

Theorem If π be a stationary distribution. Then

$$\sum_x \pi(x) P^n(x,y) = \pi(y) \quad \forall\, y \in \mathcal{S}$$

Proof

$$\sum_x \pi(x) P^{n+1}(x,y) = \sum_x \pi(x) \sum_z P^n(x,z) P(z,y)$$

$$= \sum_z \left(\sum_x \pi(x) P^n(x,z) \right) P(z,y)$$

$$= \sum \pi(z) P(z,y)$$

$$= \pi(y)$$

By induction we can get the result.

P.T.O

Theorem If X_0 has the stationary distribution π for its initial distribution, the for all n, $P(X_n = y) = \pi(y)$, $y \in \mathcal{S}$
Conversely, if distribution of X_n is independent of n then π initial distribution is a stationary distr.

Pf
$$P(X_n = y) = \sum_x \pi_0(x) \, P^n(x, y) = \pi(y)$$

Suppose conversely that the distribution of X_n is independent of n. Then the initial distribution π_0 is such that

$$\pi_0(y) = P(X_0 = y) = P(X_1 = y) = \sum_x \pi_0(x) \, P(x, y)$$

Consequently π_0 is a stationary distribution.

Hence, the distribution of X_n is independent of n iff the initial distribution is a stationary distribution.

Bounded Convergence Theorem

Let $a(x)$, $x \in \mathcal{S}$, be nonnegative numbers having finite sum, and let $b_n(x)$, $x \in \mathcal{S}$ and $n \geq 1$, be such that $|b_n(x)| \leq 1$, $x \in \mathcal{S}$, and $n \geq 1$, and

$$\lim_{n \to \infty} b_n(x) = b(x), \quad x \in \mathcal{S}$$

Then, $$\lim_{n \to \infty} \sum_x a(x)\, b_n(x) = \sum_x a(x)\, b(x)$$

Result: Suppose that π is a stationary distribution and

$$\lim_{n \to \infty} P^n(x, y) = \pi(y) \qquad y \in \mathcal{S}$$

Let π_0 be the initial distribution. Then

$$P(X_n = y) = \sum_x \pi_0(x)\, P^n(x, y) \qquad y \in \mathcal{S}$$

$$\therefore \lim_{n \to \infty} P(X_n = y) = \sum_x \pi_0(x)\, \pi(y) = \pi(y) \qquad y \in \mathcal{S}.$$

ie regardless of the initial distribution, for large values of n the distribution of X_n is approximately equal to the stationary distribution π.

Hence, π is the unique stationary distribution. For if there were some other stationary distribution we take it as the initial distribution π_0. then

$$P(X_n = y) = \pi_0(y) \qquad y \in \mathcal{S}$$

and hence, $$\pi_0(y) = \pi(y) \qquad y \in \mathcal{S}.$$

Example 1 Consider two-state Markov chain on $\mathcal{S} = \{0,1\}$ having transition matrix $\begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$ and $p+q>0$

Let, $\pi(0) = \dfrac{q}{p+q}$ and $\pi(1) = \dfrac{p}{p+q}$

Now, $\sum\limits_x \pi(x)\, P(x,0) = \dfrac{q}{p+q}\cdot(1-p) + \dfrac{p}{p+q}\cdot q$

$$= \frac{q}{p+q} = \pi(0)$$

$$\sum_x \pi(x)\, P(x,1) = \frac{q}{p+q}\cdot p + \frac{p}{p+q}(1-q) = \pi(1)$$

$\therefore$ π is a stationary distribution.

Also, $P^n = \dfrac{1}{p+q}\begin{bmatrix} q & p \\ q & p \end{bmatrix} + \dfrac{(1-p-q)^n}{p+q}\begin{bmatrix} p & -p \\ -q & q \end{bmatrix}$

$\therefore \lim\limits_{n\to\infty} P^n = \dfrac{1}{p+q}\begin{bmatrix} q & p \\ q & p \end{bmatrix}$

Hence $\lim\limits_{n\to\infty} P^n(x,y) = \pi(y)$

P.T.O

Example 2 Consider a Markov chain having state space $S = \{0, 1, 2\}$ and transition matrix

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{bmatrix}$$

If π is a stationary distribution then

$$\frac{\pi(0)}{3} + \frac{\pi(1)}{4} + \frac{\pi(2)}{6} = \pi(0)$$

$$\frac{\pi(0)}{3} + \frac{\pi(1)}{2} + \frac{\pi(2)}{3} = \pi(1)$$

~~and~~ $$\frac{\pi(0)}{3} + \frac{\pi(1)}{4} + \frac{\pi(2)}{2} = \pi(2)$$

and $$\pi(0) + \pi(1) + \pi(2) = 1$$

Solving the four linear equation we get

$$\pi(0) = \frac{6}{25}, \quad \pi(1) = \frac{2}{5}, \quad \pi(2) = \frac{9}{25}$$

the unique solution for the system of linear equation.

Hence, π as above is the unique stationary distribution.

Birth and Death chain (~~Example~~)

Consider a irreducible birth and death chain on $\{0,1,2,\cdots,d\}$ or on the nonnegative integers. ~~En~~ (ie, $d=\infty$)

$$P(x,y) = \begin{cases} q_x & y = x-1 \\ r_x & y = x \\ p_x & y = x+1 \end{cases}$$

$p_x + q_x + r_x = 1$, ~~and $r_x, p_x, q_x > 0$ $\forall x$~~

also, $p_x > 0$ for $0 \le x < d$, and $q_x > 0$ for $0 < x \le d$

~~we~~ also ~~assume that, $p_x, q_x > 0$~~ $p_d = q_0 = 0$

Supposse d is infinite. Then the system of equations

$$\sum_x \pi(x) P(x,y) = \pi(y), \qquad y \in \mathcal{S}$$

becomes $\pi(0) r_0 + \pi(1) q_1 = \pi(0)$ and

$$\pi(y-1) p_{y-1} + \pi(y) r_y + \pi(y+1) q_{y+1} = \pi(y), \qquad y \geq 1$$

Since $p_y + q_y + r_y = 1$, these equation reduce to

$$q_1 \pi(1) - p_0 \pi(0) = 0 \quad \text{and}$$

$$q_{y+1} \pi(y+1) - p_y \pi(y) = q_y \pi(y) - p_{y-1} \pi(y-1), \quad y \geq 1$$

Hence by induction.

$$q_{y+1} \pi(y+1) - p_y \pi(y) = 0, \qquad y \geq 0$$

and hence, $\pi(y+1) = \dfrac{p_y}{q_{y+1}} \pi(y), \qquad y \geq 0$

Consequently, $\pi(x) = \dfrac{p_0 p_1 \cdots p_{x-1}}{q_1 q_2 \cdots q_x} \pi(0) \qquad x \geq 1$

Set, $$\pi_x = \begin{cases} 1 & x = 0 \\ \dfrac{p_0 p_1 \cdots p_{x-1}}{q_1 q_2 \cdots q_x} & x \geq 1 \end{cases}$$

Then $\pi(x) = \pi_x \pi(0) \qquad x \geq 0$

P.T.O

Conversely, let $\pi(x) = \pi_x \pi(0)$, $x \geq 0$

then $$\sum_x \pi(x) P(x,0) = \pi(0) P(0,0) + \pi(1) P(1,0)$$
$$= \pi(0) r_0 + \pi_1 \pi(0) q_1$$
$$= \pi(0) \left\{ r_0 + \frac{p_0}{q_1} . q_1 \right\} = \pi(0)$$

for $y \geq 1$, $$\sum_x \pi(x) P(x,y) = \pi(y-1) P(y-1, y) + \pi(y) P(y,y) + \pi(y+1) P(y+1, y)$$
$$= \pi(0) \left\{ \pi_{y-1} p_{y-1} + \pi_y r_y + \pi_{y+1} q_{y+1} \right\}$$
$$= \pi(0) \left\{ \frac{p_0 p_1 \cdots p_{y-1}}{q_1 q_2 \cdots q_{y-1}} + \frac{p_0 p_1 \cdots p_{y-1}}{q_1 q_2 \cdots q_y} . r_y + \frac{p_0 \cdots p_y}{q_1 \cdots q_y} \right\}$$
$$= \pi(0) \frac{p_0 p_1 \cdots p_{y-1}}{q_1 q_2 \cdots q_y} = \pi(0) \pi_y = \pi(y)$$

Suppose now that $\sum_x \pi_x < \infty$ i.e. $\sum_{x=1}^{\infty} \frac{p_0 \cdots p_{x-1}}{q_1 \cdots q_x} < \infty$

Then, the birth and death chain has a unique stationary distribution, given by $\pi(x) = \frac{\pi_x}{\sum_{y=0}^{d} \pi_y}$ $x \geq 0$

On contrary, if $\sum_{x=0}^{d} \pi_x = \infty$ then any solution is either identically zero or has infinite sum, and hence there is no stationary distribution.

Suppose now that $d < \infty$. By essentially the same arguments ~~used to~~ we conclude that the unique stationary distribution is given by

$$\pi(x) = \frac{\pi_x}{\sum_{y=0}^{d} \pi_y} \qquad 0 \leq x \leq d$$

Where π_x, $0 \leq x \leq d$, is as in case of $d = \infty$.

Example Consider the Ehrenfest chain for $d=3$.

Then the transition matrix of the chain is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This is an irreducible birth and death chain in which

$$\pi_0 = 1, \quad \pi_1 = \frac{p_0}{q_1} = \frac{1}{1/3} = 3, \quad \pi_2 = \frac{p_0 p_1}{q_1 q_2} = \frac{1 \cdot 2/3}{1/3 \cdot 2/3} = 3$$

and $$\pi_3 = \frac{p_0 p_1 p_2}{q_1 q_2 q_3} = \frac{1 \cdot 2/3 \cdot 1/3}{1/3 \cdot 2/3 \cdot 1} = 1$$

Thus the unique stationary distribution is given by

$$\pi(0) = \frac{1}{8}, \quad \pi(1) = \frac{3}{8}, \quad \pi(2) = \frac{3}{8}, \quad \pi(3) = \frac{1}{8}$$

Example (Modified Ehrenfest chain)

Suppose we have two boxes labeled 1 and 2 and d balls labeled $1, 2, \cdots, d$. Initially some of the balls are in box 1 and remaining are in box 2. An integer is selected at random from $1, 2, \cdots, d$, and the ball labeled by that integer is removed from its box. We now select at random one of the two boxes and put the removed ball into this box. The procedure is repeated indefinitely, the selections being made independently. Let X_n denote the number of balls in box 1 after the nth trial.

Then $X_n, n \geq 0$, is a Markov chain on $\mathcal{S} = \{0, 1, \cdots, d\}$

Consider the modified Ehrenfest chain for $d=3$

$\boxed{\text{P.T.O}}$

Then the transition matrix of this chain, for $d=3$ is

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

This Markov chain is an irreducible birth and death chain.

Now, $\pi_0 = 1$

$$\pi_1 = \frac{p_0}{q_1} = \frac{1/2}{1/6} = 3$$

$$\pi_2 = \frac{p_0 p_1}{q_1 q_2} = \frac{1/2 \cdot 1/3}{1/6 \cdot 1/3} = 3$$

$$\pi_3 = \frac{p_0 p_1 p_2}{q_1 q_2 q_3} = \frac{1/2 \cdot 1/3 \cdot 1/6}{1/6 \cdot 1/3 \cdot 1/2} = 1$$

Hence $\pi(0) = \frac{1}{8}$, $\pi(1) = \frac{3}{8}$, $\pi(2) = \frac{3}{8}$, $\pi(3) = \frac{1}{8}$

is the unique stationary distribution.

Particles in a box

Suppose that ξ_n particles are added to a box at times $n = 1, 2, \ldots$ where ξ_n, $n \geq 1$, are independent and have a Poisson distribution with common parameter λ. Suppose that each particle in the box at time n, independently of all the other particles in the box and independently of how particles are added to the box, has probability $p < 1$ of remaining in the box at time $n+1$ and probability $q = 1 - p$ of being removed from the box at time $n+1$.

Let X_n denote the number of particles in the box at time n. Then X_n, $n \geq 0$, is a Markov chain.

The same Markov chain can be used to describe a telephone exchange, where ξ_n is the number of new calls starting at time n, q is the probability that a call in progress at time n terminates by time $n+1$, and X_n is the number of calls in progress at time n.

Let $R(X_n)$ denote the number of particles present at time n that remain in the box at time $n+1$. Then $X_{n+1} = \xi_{n+1} + R(X_n)$.

Clearly, $P(R(X_n) = z \mid X_n = x) = \binom{x}{z} p^z (1-p)^{x-z}$

$$0 \leq z \leq x$$

and $P(\xi_n = z) = \dfrac{\lambda^z e^{-\lambda}}{z!}$ $\quad z \geq 0$

P.T.O.

Since, $P(X_{n+1}=y \mid X_n=x) = \sum_{z=0}^{\min(x,y)} P(R(X_n)=z,\ \xi_{n+1}=y-z \mid X_n=x)$

$$= \sum_{z=0}^{\min(x,y)} P(\xi_{n+1}=y-z)\, P(R(X_n)=z \mid X_n=x)$$

$$\therefore\ P(x,y) = \sum_{z=0}^{\min(x,y)} \frac{\lambda^{y-z} e^{-\lambda}}{(y-z)!} \binom{x}{z} p^z (1-p)^{x-z}$$

Now, suppose $X_n \sim P(t)$ then

$$P(R(X_n)=y) = \sum_{x=y}^{\infty} P(X_n=x,\ R(X_n)=y)$$

$$= \sum_{x=y}^{\infty} P(X_n=x)\, P(R(X_n)=y \mid X_n=x)$$

$$= \sum_{x=y}^{\infty} \frac{t^x e^{-t}}{x!} \binom{x}{y} p^y (1-p)^{x-y}$$

$$= \sum_{x=y}^{\infty} \frac{t^x e^{-t}}{y!\,(x-y)!} p^y (1-p)^{x-y}$$

$$= \frac{(pt)^y e^{-t}}{y!} \sum_{z=0}^{\infty} \frac{(t(1-p))^z}{z!}$$

$$= \frac{(pt)^y e^{-pt}}{y!}$$

which shows that $R(X_n)$ has a Poisson distribution with parameter pt.

To find the stationary distribution let X_0 have Poisson distribution with parameter t.

Then $X_1 = \xi_1 + R(X_0)$, is the sum of independent random variables having poisson distribution with parameters λ and pt resp. Thus $X_1 \sim P(\lambda + pt)$. The distribution of X_1 will agree with that of X_0 if $t = \frac{\lambda}{1-p} = \frac{\lambda}{q}$

Hence, the Markov chain has stationary distribution π (31)

which is a Poisson distribution with parameter λ/q ;

To find the formula for $P^n(x,y)$, suppose X_0 has a Poisson distribution with parameter t, then

$X_1 \sim P(\lambda + pt)$

$X_2 \sim P(\lambda + p(\lambda + pt))$.

i.e $X_2 \sim P(\lambda(1+p) + p^2 t)$

$X_3 \sim P(\lambda + p(\lambda(1+p) + p^2 t))$

i.e. $X_3 \sim P(\lambda(1+p+p^2) + p^3 t)$

Continuing we get $X_n \sim P((1+p+\cdots+p^{n-1})\lambda + p^n t)$

i.e $X_n \sim P\left(tp^n + \frac{\lambda}{q}(1-p^n)\right)$

Thus $\sum_{x=0}^{\infty} \frac{e^{-t} t^x}{x!} P^n(x,y) = P(X_n = y)$

$$\therefore \sum_{x=0}^{\infty} \frac{e^{-t} t^x}{x!} P^n(x,y) = \exp\left[-\left(tp^n + \frac{\lambda}{q}(1-p^n)\right)\right] \frac{\left[tp^n + \frac{\lambda}{q}(1-p^n)\right]^y}{y!}$$

and hence,

$$\sum_{x=0}^{\infty} t^x \frac{P^n(x,y)}{x!} = e^{-\lambda(1-p^n)/q} e^{t(1-p^n)} \frac{\left[tp^n + \frac{\lambda}{q}(1-p^n)\right]^y}{y!}$$

We know that if $\sum_{x=0}^{\infty} c_x t^x = \left(\sum_{x=0}^{\infty} b_x t^x\right)\left(\sum_{x\geq 0}^{\infty} a_x t^x\right)$

where each power series has a positive radius of convergence, then $c_x = \sum_{z=0}^{x} a_z b_{x-z}$

If $a_z = 0$ for $z > y$, the $c_x = \sum_{z=0}^{\min(x,y)} a_x b_{x-z}$

P.T.O.

Using this result and the binomial expansion.

we get,

$$P^n(x,y) = \frac{x!\, e^{-\lambda(1-p^n)/q}}{y!} \sum_{z=0}^{\min(x,y)} \binom{y}{z} p^{nz} \left[\frac{\lambda}{q}(1-p^n)\right]^{y-z} \frac{(1-p^n)^{x-z}}{(x-z)!}$$

$$= e^{-\lambda(1-p^n)/q} \sum_{z=0}^{\min(x,y)} \binom{x}{z} p^{nz} (1-p^n)^{x-z} \frac{\left[\lambda/q\,(1-p^n)\right]^{y-z}}{(y-z)!}$$

Now, $\lim\limits_{n\to\infty} p^n = 0$ as $p<1$

$$\therefore \lim_{n\to\infty} P^n(x,y) = \frac{e^{-\lambda/q}\left(\frac{\lambda}{q}\right)^y}{y!} = \pi(y)$$

$x, y \geq 0$

Consequently the distribution π is the unique stationary distribution of the chain.

Renewal Theory

• **Def^n** If the sequence of nonnegative random variables $\{X_1, X_2, \ldots\}$ is independent and identically distributed, then the counting process $\{N(t) ; t \geq 0\}$ (i.e. $N(t)$ is the total number of "events" that occur by time t) is said to be a renewal process.

Example Suppose that we have an infinite supply of lightbulbs whose lifetimes are independent and identically distributed. Suppose also that we use a single lightbulb at a time, and when it fails we immediately replaced it ~~which~~ with a new one. Under these conditions, $\{N(t), t \geq 0\}$ is a renewal process when $N(t)$ represents the number of lightbulbs that have failed by time t.

For a renewal process having interarrival times $X_1, X_2, \ldots$ let $S_0 = 0$, $S_n = \sum_{i=1}^{n} X_i$, $n \geq 1$.

Let F denote the interarrival distribution and to avoid trivial times, we assume that $F(0) = P[X_n = 0] < 1$.

Furthermore, we let

$$\mu = E[X_n], \qquad n \geq 1$$

$\therefore \mu > 0$

Result 1 Infinite number of renewals cannot occur in a finite amount of time.

Pf $N(t) = \max\{n : S_n \le t\}$

Now by strong law of large numbers

$$P\left(\frac{S_n}{n} \to \mu \quad \text{as} \quad n \to \infty\right) = 1$$

Since $\mu > 0$, ~~$P(S_n \le t$, for any finite $n) = 1$~~

$\therefore$ $\forall$ $t > 0$, $\exists$ n_0 s.t.

$$P(S_n \le t, \quad \text{for } n > n_0) = 0$$

$\therefore$ $N(t)$ is finite with probability 1. for all finite t.

Result 2 $P(N(\infty) < \infty) = 0$ where $N(\infty) = \lim_{t \to \infty} N(t)$

Pf. $P(N(\infty) < \infty) = P(X_n = \infty$ for some $n)$

$$= P\left(\bigcup_{n=1}^{\infty} \{X_n = \infty\}\right)$$

$$\le \sum_{n=1}^{\infty} P(X_n = \infty) = 0$$

P.T.O

Distribution of $N(t)$

$$N(t) \geq n \iff S_n \leq t$$

$$P\{N(t) = n\} = P\{N(t) \geq n\} - P\{N(t) \geq n+1\}$$
$$= P\{S_n \leq t\} - P\{S_{n+1} \leq t\}$$

Since, X_i, $i \geq 1$, are i.i.d $F(\cdot)$
it follows that S_n is distributed F_n, the n-fold convolution of F with itself.

$$\therefore P(N(t) = n) = F_n(t) - F_{n+1}(t)$$

Example Suppose $P(X_n = i) = p(1-p)^{i-1}$, $i \geq 1$

That is, the interarrival distribution is geometric

Then, S_n has the negative binomial distribution

$$P(S_n = k) = \begin{cases} \binom{k-1}{n-1} p^n (1-p)^{k-n} & k \geq n \\ 0 & k < n \end{cases}$$

Thus,
$$P(N(t) = n) = \sum_{k=n}^{[t]} \binom{k-1}{n-1} p^n (1-p)^{k-n} - \sum_{k=n+1}^{[t]} \binom{k-1}{n} p^{n+1} (1-p)^{k-n-1}$$

Equivalently, since an event independently occurs with probability p at each of the time $1, 2, \ldots$

$$P[N(t) = n] = \binom{[t]}{n} p^n (1-p)^{[t]-n}$$

Mean value of the renewal function

$$m(t) = E[N(t)]$$
$$= \sum_{n=1}^{\infty} P\{N(t) \geq n\}$$
$$= \sum_{n=1}^{\infty} P(S_n \leq t)$$
$$= \sum_{n=1}^{\infty} F_n(t)$$

It can be shown that the mean value function $m(t)$ uniquely determines the renewal process. Specifically, there is a one-one correspondence between the interarrival distributions F and the mean-value functions $m(t)$.

Example Since $m(t)$ uniquely determines the interarrival distribution, it follows that the Poisson process is the only renewal process having a linear mean value function.

Hence if $m(t) = 2t$ for a renewal process then it is the mean-value function of Poisson process with rate 2.

Result $m(t) < \infty$, for all $t < \infty$

Note that, $P(Y < \infty) = 1$ does not imply $E(Y) < \infty$
Try with $P(Y = 2^n) = \left(\frac{1}{2}\right)^n$ $n \geq 1$

Renewal Equation

Assuming that the interarrival distribution F is continuous with density function f.

$$m(t) = E(N(t)) = \int_0^\infty E[N(t) \mid X_1 = x] f(x)\, dx$$

Now suppose that the first renewal occurs at a time x that is less than t.

Then, using the fact that a renewal process probabilistically starts over when a renewal occurs, it follows that the number of renewals by time t would have the same distribution as 1 plus the number of renewals in the first t − x time units. Therefore,

$$E[N(t) \mid X_1 = x] = 1 + E[N(t-x)] \qquad \text{if } x < t$$

Since, $E[N(t) \mid X_1 = x] = 0$ when $x > t$

we obtain,

$$m(t) = \int_0^t [1 + m(t-x)] f(x)\, dx$$

$$= F(t) + \int_0^t m(t-x) f(x)\, dx$$

<u>Example</u> Let the interarrival distribution is uniform(0,1)

For $t \leq 1$,

$$m(t) = t + \int_0^t m(t-x)\, dx$$

$$= t + \int_0^t m(x)\, dx$$

$$\therefore m'(t) = 1 + m(t)$$

$$\Rightarrow m(t) = Ke^t - 1$$

Since $m(0) = 0$ Therefore $m(t) = e^t - 1$, $0 \leq t \leq 1$.

Stoping Time

Let $X_1, X_2, \cdots$ be a seq^n of independent r.vs.
The nonnegative integer valued random variable N is said to be a stoping time for the sequence if the event $\{N = n\}$ is independent of $X_{n+1}, X_{n+2}, \cdots$

If X_is are observed one at a time – first X_1, then X_2, and so on and N represents the number observed when we stop. Hence the event $\{N = n\}$ corresponds to stoping after having observed $X_1, \cdots, X_n$ and thus must be independent of the values of random variables yet to come, namely, $X_{n+1}, X_{n+2}, \cdots$

Example Let $X_1, X_2, \cdots$ be independent with

$$P[X_i = 1] = p = 1 - P[X_i = 0] \qquad i \geq 1$$

a. Then $N := \min\{n : X_1 + X_2 + \cdots + X_n = 5\}$ is a stopping times for the sequence $X_1, X_2, \cdots$

B. If $N = \begin{cases} 3 & \text{if } X_1 = 0 \\ 5 & \text{if } X_1 = 1 \end{cases}$

Then N is a stopping times for the sequence $X_1, X_2, \cdots$

Wald's Equation

Let $X_1, X_2, \ldots$ are independent and identically distributed and having a finite mean $E(X)$.

and if N is a stopping time for this seq^n having a finite mean,

Define ~~that~~ the indicator variables I_i, $i \geq 1$

by
$$I_i = \begin{cases} 1 & \text{if } i \leq N \\ 0 & \text{if } i > N \end{cases}.$$

$$\therefore \sum_{i=1}^{N} X_i = \sum_{i=1}^{\infty} X_i I_i$$

$$\therefore E\left[\sum_{i=1}^{N} X_i\right] = \sum_{i=1}^{\infty} E\left[X_i I_i\right]$$

$$= \sum_{i=1}^{\infty} E[X]\, E[I_i]$$

$(\because X_i$ and I_i are independent$)$

$$= E[X]\, E\left[\sum_{i=1}^{\infty} I_i\right]$$

$$= E[X]\, E[N].$$

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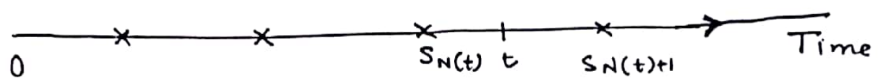
Limit theorems

Proposition $\dfrac{N(t)}{t} \longrightarrow \dfrac{1}{\mu}$ as $t \to \infty$ with prob 1.

Proof. Consider the random variable $S_{N(t)}$, the time of the last renewal prior to ~~the~~ or at time t.

Obiviously, $S_{N(t)+1}$ represents the time of the first renewal after time t.



$$\therefore \quad \frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} \leq \frac{S_{N(t)+1}}{N(t)}$$

However, since $\dfrac{S_{N(t)}}{N(t)} = \dfrac{\sum_{i=1}^{N(t)} X_i}{N(t)}$ is the average of $N(t)$ independent and identically distributed random variables, it follows by the strong law of large numbers that $S_{N(t)}/N(t) \longrightarrow \mu$ as $N(t) \to \infty$ with prob 1

But since $N(t) \to \infty$ when $t \to \infty$ "

we obtain $\dfrac{S_{N(t)}}{N(t)} \longrightarrow \mu$ as $t \longrightarrow \infty$ "

Furthermore,

$$\frac{S_{N(t)+1}}{N(t)} = \left(\frac{S_{N(t)+1}}{N(t)+1}\right)\left(\frac{N(t)+1}{N(t)}\right) \to \mu \cdot 1 = \mu$$

as $t \to \infty$ with prob 1

Hence the proposition.

P. T. O

Def^n

The number $1/\mu$ is called the rate of the renewal process.

Elementary Renewal theorem

$$\frac{m(t)}{t} \longrightarrow \frac{1}{\mu} \quad \text{as } t \to \infty.$$

if μ is finite

and $$\frac{m(t)}{t} \longrightarrow 0 \quad \text{as } t \to \infty$$

if μ is infinite.

Pf Suppose that $\mu < \infty$

then $S_{N(t)+1} > t$

Let $X_1, X_2, \ldots$ denote the interarrival times of a renewal process and let us stop at the first renewal after t, i.e., at the $N(t)+1$ renewal. To verify that $N(t)+1$ is a stopping time for X_i, $i=1,2,\ldots$

note that $N(t)+1 = n \iff N(t) = n-1$

$\iff X_1 + \cdots + X_{n-1} \leq t$

and $X_1 + \cdots + X_n > t$

Thus the event $\{N(t)+1 = n\}$ depends only on $X_1, \ldots X_n$ and is thus independent of $X_{n+1}, \ldots$ hence $N(t)+1$ is a stopping time

From the Wald's equation we obtain that,

when $E[X] < \infty$,

$$E[X_1 + X_2 + \cdots + X_{N(t)+1}] = E[X]\, E[N(t)+1]$$

equivalently,

if $\mu < \infty$ then $E[S_{N(t)+1}] = \mu\,[m(t)+1]$

Since $S_{N(t)+1} > t$

$\therefore \quad \mu \cdot (m(t)+1) > t$

$$\Rightarrow \liminf_{t \to \infty} \frac{m(t)}{t} \geq \frac{1}{\mu}$$

To go the other way, we fix a constant M, and define a new renewal process $\{\bar{X}_n, n = 1, 2, \ldots\}$ by letting

$$\bar{X}_n = \begin{cases} X_n & \text{if } X_n \leq M \\ M & \text{if } X_n > M \end{cases} \qquad n = 1, 2, \ldots$$

Let $\bar{S}_n = \sum_{i=1}^{n} \bar{X}_i$ and $\bar{N}(t) = \sup\{n : \bar{S}_n \leq t\}$.

Since the interarrival times for this truncated renewal process are bounded by M, we have

$$\bar{S}_{\bar{N}(t)+1} \leq t + M$$

Hence $(\bar{m}(t) + 1)\,\mu_M \leq t + M$

where $\mu_M = E[\bar{X}_n]$

Thus $\limsup_{n \to \infty} \frac{\bar{m}(t)}{t} \leq \frac{1}{\mu_M}$

Now, since $\bar{S}_n \le S_n$, it follows that $\bar{N}(t) \ge N(t)$

and $\bar{m}(t) \ge m(t)$, thus

$$\limsup_{t\to\infty} \frac{m(t)}{t} \le \frac{1}{\mu_M}$$

letting $M \to \infty$ yields

$$\limsup_{t\to\infty} \frac{m(t)}{t} \le \frac{1}{\mu}$$

Hence the results.

When $\mu = \infty$, we consider the truncated process;

Since $\mu_M \to \infty$ as $M \to \infty$

the results follows from the fact that

$$\limsup_{t\to\infty} \frac{m(t)}{t} \le \frac{1}{\mu_M}$$

<u>Theorem</u> Let μ and $\sigma^2 < \infty$, represents the mean and variance of an interarrival time. Then

$$P\left\{ \frac{N(t) - t/\mu}{\sigma\sqrt{t/\mu^3}} < y \right\} \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2}\,dx \quad \text{as } t \to \infty$$

<u>Pf</u> Let $r_t = t/\mu + y\sigma\sqrt{t/\mu^3}$

Then $$P\left\{ \frac{N(t) - t/\mu}{\sigma\sqrt{t/\mu^3}} < y \right\} = P\left(N(t) < r_t\right)$$

$$= P\left(S_{r_t} > t\right)$$

$$\left(\because N(t) < n \Leftrightarrow S_n > t\right)$$

$$= P\left\{ \frac{S_{r_t} - r_t\mu}{\sigma\sqrt{r_t}} > \frac{t - r_t\mu}{\sigma\sqrt{r_t}} \right\}$$

$$= P\left\{ \frac{S_{r_t} - r_t\mu}{\sigma\sqrt{r_t}} > -y\left(1 + \frac{y\sigma}{\sqrt{t\mu}}\right)^{-1/2} \right\}$$

Now, by the CLT, $(S_{r_t} - r_t\mu)/\sigma\sqrt{r_t}$ converges to a normal random variable having mean 0 and variance 1 as $t \to \infty$ (thus $r_t \to \infty$).

Also since, $-y\left(1 + \frac{y\sigma}{\sqrt{t\mu}}\right)^{-1/2} \longrightarrow -y$ as $t \to \infty$.

we see that $$P\left\{ \frac{N(t) - t/\mu}{\sigma\sqrt{t/\mu^3}} < y \right\} \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-y}^{\infty} e^{-x^2/2}\,dx$$

and since $$\int_{-y}^{\infty} e^{-x^2/2}\,dx = \int_{-\infty}^{y} e^{-x^2/2}\,dx$$

the result follows.

Lattice

A nonnegative random variable X is said to be lattice if there exists $d \geq 0$ such that $\sum_{n=0}^{\infty} P(X = nd) = 1$

That is, X is lattice if it only takes on integral multiples of some nonnegative number d.

The largest d having this property is said to be the period of X. If X is lattice and F is the distribution function of X, then we say that F is lattice.

Blackwell's theorem

i) If F is not lattice, then

$$m(t+a) - m(t) \rightarrow a/\mu \quad \text{as } t \rightarrow \infty$$

for all $a \geq 0$.

ii) If F is a lattice with period d, then

$$E[\text{number of renewals at } nd] \rightarrow d/\mu \quad \text{as } n \rightarrow \infty$$

Key Renewal theorem

If F is not lattice and if $h(t)$ is directly Riemann integrable, then $\lim_{t \to \infty} \int_0^t h(t-x)\,dm(x) = \frac{1}{\mu} \int_0^t h(t)\,dt$

Where $m(x) = \sum_{n=1}^{\infty} F_n(x)$ and $\mu = \int_0^{\infty} \overline{F}(t)\,dt$

SECOND Order Process

A stochastic process $X(t)$, $t \in T$. is called a second order process if $EX^2(t) < \infty$, $\forall\ t \in T$.

Let $X(t)$, $t \in T$, be a second order process.
The mean function $\mu_X(t)$, $t \in T$, of the process is defined by $\mu_X(t) = EX(t)$
The covariance function or auto-covariance function $r_X(s,t)$, $s,t \in T$, is defined by

$$r_X(s,t) = Cov(X(s), X(t))$$

$$\therefore\ Var(X(t)) = r_X(t,t), \quad t \in T$$

$r_X(s,t)$ is symmetric as $r_X(s,t) = r_X(t,s)$.

also, $r_X(s,t)$ is nonnegative definite.
Since, if n is a positive integer, $t_1, t_2, \dots, t_n \in T$ and $b_1, b_2, \dots, b_n \in \mathbb{R}$, then

$$\sum_{i=1}^{n} \sum_{j=1}^{n} b_i b_j\, r_X(t_i, t_j) = Var\left(\sum_{i=1}^{n} b_i X(t_i)\right) \geq 0$$

Second Order stationary process

We say that, $X(t)$, $-\infty < t < \infty$, is a second order stationary process if for every number τ the second order process $Y(t)$, $-\infty < t < \infty$, defined by $Y(t) = X(t+\tau)$ $-\infty < t < \infty$ has the same mean and covariance function as the $X(t)$ process.

P.T.O

Theorem $X(t)$ is a second order stationary process iff $\mu_X(t)$ is independent of t and $\gamma_X(s,t)$ depends only on the difference between s and t.

Pf let $X(t)$ be a second order stationary process

then $E(Y(t)) = E(X(t+\tau)) = E(X(t)) \ \forall\ t, \tau \in \mathbb{R}$.

$\therefore\ EY(0) = EX(\tau) = E(X(0)) \quad \forall \quad \tau \in \mathbb{R}$

i.e. $\quad E X(t) = E(X(0)) \qquad \forall\ t \in \mathbb{R}$

$\therefore \quad \mu_X(t) = \mu_X(0) \qquad \forall\ t \in \mathbb{R}$

Also, $\quad \mathrm{Cov}(Y(s), Y(t)) = \mathrm{Cov}(X(s), X(t)) \qquad \forall\ s, t, \tau \in \mathbb{R}$

$\therefore\ \mathrm{Cov}(X(s+\tau), X(t+\tau)) = \mathrm{Cov}(X(s), X(t)) \qquad \forall\ s, t \in \mathbb{R}$

$\therefore\ \mathrm{Cov}(X(0), X(t-s)) = \mathrm{Cov}(X(s), X(t))$

$\therefore\ \gamma_X(0, t-s) = \gamma_X(s,t) \qquad \forall\ s, t \in \mathbb{R}$.

Coversely, let $\mu_X(t)$ is independent of t $\quad \forall\ t, \tau \in \mathbb{R}$

$\therefore\ \cancel{\mu_X(t) = \mu_X(0)}$ $\quad \therefore\ \mu_X(t+\tau) = \mu_X(t)$

$\therefore\ \cancel{\mu_X(t+\mu) = \mu}$ $\quad \therefore\ E(X(t+\tau)) = E(X(t)) \ \forall\ t, \tau$

also let $\gamma_X(s,t)$ depends only on the difference between s and t.

$\therefore\ \cancel{\gamma_X(s,t) = \gamma_X(0, t-s)}$

$\therefore\ \gamma_X(s+\tau, t+\tau) = \gamma_X(s,t)$

$\therefore\ \mathrm{Cov}(X(s+\tau), X(t+\tau)) = \mathrm{Cov}(X(s), X(t))$

$\forall\ s, t \in \mathbb{R}$.

Results (second order stationary process).

Let $X(t)$, $-\infty < t < \infty$, be a second order stationary process. Then $\mu_X(t) = \mu_X(0) = \mu_X$ (say)

and $\gamma_X(s,t) = \gamma_X(0, t-s) = \gamma_X(t-s)$ (say)

where $\gamma_X(t) = \gamma_X(0,t)$

Then i) $\gamma_X(-t) = \gamma_X(t)$ $\quad \forall\, t \in \mathbb{R}$.

ii) $Var\, X(t) = \gamma_X(t,t) = \gamma_X(0)$ $\quad \forall\, t \in \mathbb{R}$

iii) $(cov(X(0), X(t)))^2 \leq Var(X(0))\, Var(X(t))$

i.e. $|\gamma_X(t)| \leq \gamma_X(0)$ $\quad \forall\, t \in \mathbb{R}$

iv) If $\gamma_X(0) > 0$ then

$$cor(X(s), X(s+t)) = \frac{\gamma_X(s, s+t)}{\sqrt{\gamma_X(0)\,\gamma_X(0)}} = \frac{\gamma_X(t)}{\gamma_X(0)} \quad \forall\, t \in \mathbb{R}$$

Example 1 Let Z_1 and Z_2 be independent normally distributed random variables each having mean 0 and variance σ^2. Let λ be a real constant and set

$X(t) = Z_1 \cos \lambda t + Z_2 \sin \lambda t$, $\quad -\infty < t < \infty$.

Now, $\mu_X(t) = E Z_1 \cos \lambda t + E Z_2 \sin \lambda t = 0$ $\quad \forall\, t \in \mathbb{R}$

and $\gamma_X(s,t) = cov(X(s), X(t))$

$= E[X(s)\, X(t)]$

$= E[Z_1^2 \cos \lambda s \cos \lambda t + Z_2^2 \sin \lambda s \sin \lambda t]$

$= \sigma^2 (\cos \lambda s \cos \lambda t) + \sigma^2 (\sin \lambda s \sin \lambda t)$

$= \sigma^2 \cos \lambda (t-s)$

$\therefore \gamma_X(t) = \sigma^2 \cos \lambda t$

This shows that $X(t)$ is a second order stationary process.

P.T.O.

Cross - covariance Function

Consider two second order processes $X(t)$, $t \in T$, and $Y(t)$, $t \in T$. Their cross - covariance function is defined as $\gamma_{XY}(s,t) = Cov(X(s), Y(t))$ $\quad s, t \in T$

Clearly $\gamma_{XY}(s,t) = \gamma_{YX}(t,s)$

and $\gamma_{XX}(s,t) = \gamma_X(s,t)$

Now, $\gamma_{X+Y}(s,t) = \gamma_X(s,t) + \gamma_{XY}(s,t) + \gamma_{YX}(s,t) + \gamma_Y(s,t)$

if cross covariance function vanishes then

$\gamma_{X+Y}(s,t) = \gamma_X(s,t) + \gamma_Y(s,t)$

$\therefore \gamma_{X+Y}(t) = \gamma_X(t) + \gamma_Y(t) \quad \forall\ t \in T$

Now, Consider n second order stationary processes $X_1(t), X_2(t), \ldots, X_n(t)$ $\quad -\infty < t < \infty$, whose cross - covariance functions all vanish.

Then their sum
$$X(t) = X_1(t) + \cdots + X_n(t) \quad -\infty < t < \infty$$
is a second order stationary process such that
$$\mu_X = \sum_{k=1}^{n} \mu_{X_k}$$
and
$$\gamma_X(t) = \sum_{k=1}^{n} \gamma_{X_k}(t) \quad -\infty < t < \infty$$

Example Let $Z_{11}, Z_{12}, Z_{21}, Z_{22}, \ldots, Z_{n1}, Z_{n2}$ be $2n$ independent normally distributed random variables each having mean zero and such that
$$Var\, Z_{k1} = Var\, Z_{k2} = \sigma_k^2 \quad k = 1, 2, \ldots n$$
Let $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ and set
$$X(t) = \sum_{k=1}^{n} (Z_{k1} \cos \lambda_k t + Z_{k2} \sin \lambda_k t) \quad t \in \mathbb{R}$$

P.T.O

Set $X_k(t) = Z_{k1} \cos \lambda_k t + Z_{k2} \sin \lambda_k t$

Since, Z's are independent, the cross-covariance function between $X_i(t)$ and $X_j(t)$ vanishes for $i \neq j$

Now, $X_k(t)$'s are second order stationary process having mean zero and covariance function $r_{X_k}(t) = \sigma_k^2 \cos \lambda_k t$

$\therefore$ $X(t)$, $-\infty < t < \infty$, is a second order stationary process having mean zero and covariance function

$$r_X(t) = \sum_{i=1}^{n} \sigma_k^2 \cos \lambda_k t, \quad -\infty < t < \infty$$

Gaussian Processes

A stochastic process $X(t)$, $t \in T$ is called a Gaussian process if every finite linear combination of the random variables $X(t)$, $t \in T$, is normally distributed. (Constant random variables are regarded as normally distributed with zero variance.)

Gaussian processes are also called normal processes, and normally distributed random variables are said to ~~be~~ have a Gaussian distribution.

If $X(t)$, $t \in T$ is a Gaussian process, then for each $t \in T$, $X(t)$ is normally distributed and $E X^2(t) < \infty$. Thus a Gaussian process is a second order process.

Example The process $X(t)$, $-\infty < t < \infty$ where

$$X(t) = Z_1 \cos \lambda t + Z_2 \sin \lambda t$$

as in previous example (example for second order process) is a Gaussian process

Also, ~~exam~~ Z_{k1},

$$X(t) = \sum_{k=1}^{n} \left(Z_{k1} \cos \lambda_k t + Z_{k2} \sin \lambda_k t \right)$$

$-\infty < t < \infty$, as in previous example is a Gaussian process.

Definition

Two stochastic process $X(t)$, $t \in T$ and $Y(t)$, $t \in T$, are said to have the same joint distribution functions if for every positive integer n and every choice of $t_1, \dots, t_n \in T$, the random variables $X(t_1), \dots, X(t_n)$ have the same joint distribution function as the random variables. $Y(t_1), \dots, Y(t_n)$.

Theorem If two Gaussian processes have the same mean and covariance functions, then they also have the same joint distribution functions.

Pf Omitted.

Example Let $X(t)$, $t \in T$ be a Gaussian process having zero means.

Now, $X(t) \sim N(0, r_X(t,t))$

$$\therefore \; E X^4(t) = 3(r_X(t,t))^2$$

Definition Let $X_1, \dots, X_n$ be n random variables. They are said to have a joint normal or Gaussian distribution, if $a_1 X_1 + a_2 X_2 + \dots + a_n X_n$ is normally distributed for every choice of the constants $a_1, a_2, \dots, a_n$.

(P.T.O)

A stochastic process $X(t)$, $t \in T$, is a Gaussian process iff for every positive integer n and every choice of $t_1, t_2, \ldots t_n \in T$, the random variables $X(t_1), \ldots, X(t_n)$ have a joint normal distribution.

Let $X_1, \ldots, X_n$ be random variables having a joint normal distribution and a density f ~~and~~ to (such a density exists iff the covariance matrix of $X_1, X_2, \ldots X_n$ has nonzero determinant).

Then $$f(x_1, \ldots x_n) = \frac{1}{(2\pi)^{n/2} (\det \Sigma)^{1/2}} \exp\left[-\tfrac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\right]$$

where, $$\Sigma = \begin{bmatrix} \operatorname{cov}(X_1, X_1) & \cdots & \operatorname{cov}(X_1, X_n) \\ \vdots & & \vdots \\ \operatorname{cov}(X_n, X_1) & \cdots & \operatorname{cov}(X_n, X_n) \end{bmatrix}$$

$$x = (x_1, \ldots, x_n)' , \quad \mu = (\mu_1, \ldots, \mu_n)'$$

Also, the conditional expectation of X_n given $X_1, \ldots, X_{n-1}$ is a linear function of these $n-1$ random variables.

i.e. $E\left[X_n \mid X_1 = x_1, \ldots, X_{n-1} = x_{n-1}\right] = a + b_1 x_1 + \cdots + b_{n-1} x_{n-1}$

for suitable constants $a, b_1, \ldots, b_{n-1}$

Definition

A stochastic process $X(t)$, $-\infty < t < \infty$, is said to be strictly stationary if for every number τ the stochastic process $Y(t)$, $-\infty < t < \infty$ defined by $Y(t) = X(t+\tau)$ has the same joint distribution functions as the $X(t)$ process.

A strictly stationary process need not have finite second moments and hence need not be a second order process, however, if a strictly stationary process does have finite second moments, then it is a second order stationary process. But the converse is not true in general.

Let $X(t)$, $-\infty < t < \infty$, be a second order stationary process which is also a Gaussian process. Then this process is necessarily strictly stationary. For if τ is any real number, then the $Y(t)$ process defined by $Y(t) = X(t+\tau)$, $-\infty < t < \infty$ is a Gaussian process having the same mean and covariance functions as the $X(t)$ process. It therefore has the same joint distribution functions as the $X(t)$ process.

The Wiener process

Let the location of a ~~particate~~ particle be described by a Cartesian coordinate system whose origin is the location of the particle at time $t=0$. Then the three coordinates of the position of the particle vary independently, each according to a stochastic process $W(t)$, $-\infty < t < \infty$, satisfying the following properties.

i) $W(0) = 0$

ii) $W(t) - W(s)$ has a normal distribution with mean 0 and variance $\sigma^2(t-s)$ for $s \le t$.

iii) $W(t_2) - W(t_1)$, $W(t_3) - W(t_2)$, ..., $W(t_n) - W(t_{n-1})$ are independent for $t_1 \le t_2 \le \dots \le t_n$.

The above stochastic process $W(t)$, $-\infty < t < \infty$, is called the Wiener process with parameter σ^2.

Clearly, $W(t) = W(t) - W(0) \sim N(0, \sigma^2 t)$ for $t \geq 0$

and $E\{(W(t_2) - W(t_1)\}(W(t_4) - W(t_3)) = 0$

for $t_1 \le t_2 \le t_3 \le t_4$.

also, ~~and~~ $-W(t) = W(0) - W(t) \sim N(0, -\sigma^2 t)$ for $t<0$

$\therefore W(t) \sim N(0, -\sigma^2 t)$ for $t<0$

$\therefore W(t) \sim N(0, \sigma^2 |t|)$ for $t \in \mathbb{R}$.

P.T.O.

Theorem The covariance function of the process is

$$\gamma_W(s,t) = \begin{cases} \sigma^2 \min(|s|,|t|) & st>0 \\ 0 & st \le 0 \end{cases}$$

Pf Let $0 < s \le t$

then $\mathrm{cov}(W(s) - W(0), W(t) - W(s)) = 0$

$\Rightarrow \mathrm{cov}(W(s), W(t)) - \mathrm{Var}(W(s)) = 0$

$\Rightarrow \gamma_W(s,t) = \sigma^2 s = \sigma^2 |s|$

Similarly if $0 < t \le s$ then, $\gamma_W(s,t) = \gamma_W(t,s)$
$= \sigma^2 t = \sigma^2 |t|$

Now let, $s \le t < 0$

then $\mathrm{Cov}(W(t) - W(s), W(0) - W(t)) = 0$

$\Rightarrow -\mathrm{cov}(W(t), W(t)) + \mathrm{cov}(W(s), W(t)) = 0$

$\Rightarrow \gamma_W(s,t) = \mathrm{Var}(W(t)) = \sigma^2 |t|$

Similarly, for $t \le s < 0$
$\gamma_W(s,t) = \sigma^2 |s|$

Now let, $s \le 0 \le t$

$\mathrm{cov}(W(0) - W(s), W(t) - W(0)) = 0$

$\Rightarrow -\mathrm{cov}(W(s), W(t)) = 0$

$\Rightarrow \gamma_W(s,t) = 0$

Similarly for $t \le 0 \le s$
$\gamma_W(s,t) = 0$

Theorem The Wiener process is a Gaussian process.

Pf We have to show,

if $t_1 \le t_2 \le \dots \le t_n$ and

$b_1, b_2, \dots b_n$ are real constants,

the random variable

$b_1 W(t_1) + \dots + b_n W(t_n)$ is normally distributed.

Without loss of generality, we can assume that one of the numbers $t_1, \dots, t_n$ say t_k equals zero. Otherwise we can add one random variable $W(0)$ to the above expression and still the expression will be same.

Then each of the random variables $W(t_1), \dots W(t_n)$ is a linear combination of the increments $W(t_2) - W(t_1), \dots, W(t_n) - W(t_{n-1})$. Indeed,

$$W(t_k) = 0$$

$$W(t_j) = (W(t_{k+1}) - W(t_k)) + (W(t_{k+2}) - W(t_{k+1})) + (W(t_{k+3}) - \dots + (W(t_{j-1}) - W(t_{j-2})) + (W(t_j) - W(t_{j-1}))$$

for $k < j \le n$

and $$W(t_j) = (W(t_j) - W(t_{j+1})) + \dots + (W(t_{k-1}) - W(t_k))$$

for $1 \le j < k$

Thus $b_1 W(t_1) + b_2 W(t_2) + \dots + b_n W(t_n)$ can also be written as a linear combination of the increments $W(t_2) - W(t_1), \dots, W(t_n) - W(t_{n-1})$. Since these increments are independent and normally distributed any linear combination of them also normally distributed.

Exercise Show that

$$E\,(W(s) - W(a))(W(t) - W(a)) = \sigma^2 \min(s-a, t-a)$$

for, $s, t \geq a$

Soln Let, $s \geq t \geq a$

$$E\,(W(s) - W(a))(W(t) - W(a))$$

$$= E\,(W(s) - W(t))(W(t) - W(a)) + E\,(W(t) - W(a))(W(t) - W(a))$$

$$= 0 + var\,(W(t) - W(a))$$

$$\cancel{= Var\,(W(t)) + Var\,(W(a)) - 2\,cov\,(W(t), W(a))}$$

$$= \sigma^2 (t-a) = \sigma^2 \min(s-a, t-a)$$

Similarly for $t \geq s \geq a$.

$$E\,(W(s) - W(a))(W(t) - W(a))$$

$$= E(W(s) - W(a))(W(t) - W(s) + W(s) - W(a))$$

$$= E(W(s) - W(a))(W(t) - W(s)) + E(W(s) - W(a))(W(s) - W(a))$$

$$= 0 + var\,(W(s) - W(a))$$

$$= \sigma^2 (s-a)$$

$$= \sigma^2 \min(s-a, t-a).$$

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Brownian Motion (A different approach)

Let us start by considering the symmetric random walk that in each time unit is equally likely to take a unit step either to the left or to the right. Now suppose that we speed up this process by taking smaller and smaller steps in smaller and smaller time intervals. If we now go to the limit, what we obtain is Brownian Motion.

More precisely suppose that each Δt time units we take a step of size Δx either to the left or to the right with equal probabilities. If we let $X(t)$ denote the position at time t, then

$$X(t) = \Delta x \left(X_1 + X_2 + \cdots + X_{[t/\Delta t]} \right)$$

where

$$X_i = \begin{cases} 1 & \text{if the } i\text{th step of length } \Delta x \text{ is to the right} \\ -1 & \text{if is to the left.} \end{cases}$$

and where the X_i are assumed independent with

$$P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$$

Since $E[X_i] = 0$, $Var[X_i] = 1$.

we see that, $E(X(t)) = 0$

$$V(X(t)) = (\Delta x)^2 \left[\frac{t}{\Delta t} \right].$$

P.T.O.

We shall now let Δx and Δt go to 0. However, we must do it in a way to keep the resulting limit process nontrivial (for instance, if we let $\Delta x = \Delta t$ and then let $\Delta t \to 0$, then from the above we see that $E[X(t)]$ and $Var(X(t))$ would both converge to 0 and thus $X(t)$ would equal 0 with probability 1.). If we let $\Delta x = c\sqrt{\Delta t}$ for some positive constant c, then as $\Delta t \to 0$

$$E[X(t)] = 0 \quad , V(X(t)) \to c^2 t.$$

We now state some intuitive properties of this limiting process obtained by taking $\Delta x = c\sqrt{\Delta t}$ and then letting $\Delta t \to 0$.

Now by CLT we see that,

(i) $X(t)$ is normal with mean 0 and variance $c^2 t$.

In addition, as the changes of value of the random walk in nonoverlapping time intervals are independent, we have

(ii) $\{X(t), t \geq 0\}$ has independent increaments.

Finally, as the distribution of the change in position of the random walk over any time interval depends only on the length of that interval, it would appear that:

(iii) $\{X(t), t \geq 0\}$ has stationary increments.

Brownian Motion

A stochaotic process $\{X(t), t \geq 0\}$ is said to be a Brownian motion process if:

(i) $X(0) = 0$

(ii) $\{X(t), t \geq 0\}$ has stationary independent increments;

(iii) for every $t > 0$, $X(t)$ is normally distributed with mean 0 and variance $c^2 t$.

When $c = 1$, the process is called standard Brownian motion. Any Brownian motion can always be converted to the standard Brownian motion by looking at $X(t)/c$.

The independent increment assumption implies that the change in position between time points s and $s+t$ i.e., $X(t+s) - X(s)$ is independent of all process values before time s. Hence,

$$P\{X(t+s) \leq a \mid X(s) = x, X(u), 0 \leq u < s\}$$

$$= P\{X(t+s) - X(s) \leq a - x \mid X(s) = x, X(u), 0 \leq u < s\}$$

$$= P\{X(t+s) - X(s) \leq a - x\}$$

$$= P\{X(t+s) \leq a \mid X(s) = x\}$$

P.T.O.

which states that the conditional distribution of a future state $X(t+s)$ given the present $X(s)$ and the past $X(u)$, $0 \leq u < s$, depends only on the present. So it is a Markov process.

Since $X(t)$ is normal with mean 0 and variance t, its density function is given by

$$f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}.$$

From the stationary independent increment assumption, it easily follows that the joint density $X(t_1), X(t_2), \ldots, X(t_n)$ is given by

$$f(x_1, x_2, \ldots x_n) = f_{t_1}(x_1) f_{t_2-t_1}(x_2-x_1) \ldots f_{t_n-t_{n-1}}(x_n - x_{n-1})$$

The conditional density of $X(s)$ given that $X(t) = B$, where $s<t$, is

$$f_{s|t}(x|B) = \frac{f_s(x) f_{t-s}(B-x)}{f_t(B)}$$

$$= K_1 \exp\left\{\frac{-x^2}{2s} - \frac{(B-x)^2}{2(t-s)}\right\}$$

$$= K_2 \exp\left\{-\frac{t(x - Bs/t)^2}{2s(t-s)}\right\}$$

$\therefore$ for $s<t$, $E[X(s) \mid X(t) = B] = \frac{Bs}{t}$

and $Var(X(s) \mid X(t) = B) = s(t-s)/t$.

Markov Pure Jump process

Consider a system in state x_0 at time 0. We suppose that the system remains in state x_0 until some positive time τ_1, at which time the system jumps to a new state $x_1 \neq x_0$. We allow the possibility that the system remains permanently in state x_0, in which case we set $\tau_1 = \infty$. If τ_1 is finite, upto reaching x_1 the system remains there until some time $\tau_2 > \tau_1$ when it jumps to state $x_2 \neq x_1$. If the system never leaves x_1, we set $\tau_2 = \infty$. This procedure is repeated indefinitely. If some $\tau_m = \infty$, we set $\tau_n = \infty$ for $n > m$.

Let $X(t)$ denotes the state of the system at time t, defined by

$$X(t) = \begin{cases} x_0 & 0 \leq t < \tau_1 \\ x_1 & \tau_1 \leq t < \tau_2 \\ \vdots & \vdots \end{cases}$$

This process is called a jump process.

Note that in this process $X(t)$ may not be defined for all $t \geq 0$.

For example, consider a ball bouncing on the floor. Let the state of the system be the number of bounces it has made. We make the physically reasonable assumption that the time in seconds between the nth bounce and the $(n+1)$th bounch is 2^{-n}. Then $x_n = n$ and

PTO.

$$\tau_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}.$$

We see that $\tau_n < 2$ and $\tau_n \to 2$ as $n \to \infty$.

Thus $X(t)$ defines only for $0 \le t < 2$.

By the time $t = 2$ the ball will have made an infinite number of bounces.

In this case it would be appropriate to define $X(t) = \infty$ for $t \geq 2$.

In general, if $\lim_{n \to \infty} \tau_n < \infty$

we say that the $X(t)$ process explodes.

If the $X(t)$ process does not explode then $X(t)$ defined for all $t \geq 0$.

We suppose that all states are of one of two types, absorbing or non-absorbing.

Once the process reaches an absorbing state, it remains there permanently. With each non-absorbing state x, there is associated a distribution fn. $F_x(t)$, $-\infty < t < \infty$, which vanishes for $t \le 0$, and transition probabilities Q_{xy}, $y \in \mathcal{S}$, which are nonnegative and such that $Q_{xx} = 0$ and $\sum_y Q_{xy} = 1$.

A process starting at x remains there for a random length of time τ_1 having distribution function F_x and then jumps to state $X(\tau_1) = y$ with probability Q_{xy}, $y \in \mathcal{S}$. We assume that τ_1 and $X(\tau_1)$ are chosen independently of each other, i.e., that

$$P_x(\tau_1 \le t, X(\tau_1) = y) = F_x(t)\, Q_{xy} .$$

Here, ~~as~~ $P_x(\ )$ and $E_x(\ )$ denote probabilities of events and expectations of random variables defined in terms of a process initially in state x. Whenever and however the process jumps to a state y, it acts just as a process starting initially at y. For example, if x and y are both non-absorbing states,

$$\cancel{P_x(\tau_1 \le t, X(\tau_1) = y) = F_x(t)\, Q_{xy}}$$

$$P_x(\tau_1 \le s, X(\tau_1) = y, \tau_2 - \tau_1 \le t, X(\tau_2) = z) = F_x(s)\, Q_{xy}\, F_y(t)\, Q_{yz}$$

Similar formulas hold for events defined in terms of three or more jumps.

If x is an absorbing state, we set $Q_{xy} = \delta_{xy}$

where, $\delta_{xy} = \begin{cases} 1 & y = x , \\ 0 & y \ne x . \end{cases}$

Hence $\sum_y Q_{xy} = 1$ for all $x \in \mathcal{S}$.

We say that the jump process is pure or non-explosive if $\lim_{n\to\infty} \tau_n = \infty$ with probability one regardless of the starting point. Otherwise we say the process is explosive. If the state space $\mathcal{S}$ is finite, the jump process is necessarily non-explosive. ~~Such processes, however, are unlikely to arise~~ It is easy to construct examples having an infinite state space which are explosive.

To keep matters simple we assume that our process is non-explosive. The set of probability zero where $\lim_{n\to\infty} \tau_n \neq \infty$ can safely be ignored.

We see that $X(t)$ is then defined for all $t \geq 0$.

Let $P_{xy}(t)$ denote the probability that a process starting in state x will be in state y at time t.

Then $P_{xy}(t) = P_x(X(t) = y)$

and $\sum_y P_{xy}(t) = 1$

In particular, $P_{xy}(0) = \delta_{xy}$.

We can also chose the initial state x according to an initial distribution $\pi_0(x)$, $x \in \mathcal{S}$, where $\pi_0(x) \geq 0$

and $\sum_x \pi_0(x) = 1$.

In this case, $P(X(t) = y) = \sum_x \pi_0(x) P_{xy}(t)$.

The transition function $P_{xy}(t)$ cannot be used directly to obtain such probabilities as

$$P(X(t_1)=x_1, \ldots, X(t_n)=x_n)$$

unless the jump process satisfies the Markov property, which state that for $0 \le s_1 \le \cdots \le s_n \le s \le t$ and $x_1, x_2, \ldots, x_n, x; y \in \mathcal{S}$,

$$P(X(t)=y \mid X(s_1)=x_1, \ldots, X(s_n)=x_n, X(s)=x) =$$

$$= P_{xy}(t-s)$$

By a Markov pure jump process we mean a pure process that satisfies the Markov property.

It can be shown, ~~although~~ that a pure jump process is Markovian iff all non-absorbing states x are such that $P_x(\tau_1 > t+s \mid \tau_1 > s) = P_x(\tau_1 > t)$; $s, t \geq 0$,

i.e. such that $$\frac{1-F_x(t+s)}{1-F_x(s)} = 1-F_x(t), \quad s, t \geq 0$$

i.e. iff F_x is an exponential distribution function.

So we conclude that a pure jump is Markovian iff F_x is an exponential distribution for all non-absorbing states x.

Let $X(t)$, $0 \le t < \infty$, be a Markov pure jump process. If x is a non-absorbing state, then F_x has an exponetial density f_x. Let q_x denote the parameter of this density. Then $q_x = \frac{1}{E_x(\tau_1)} > 0$ and

$$f_x(t) = \begin{cases} q_x e^{-q_x t}, & t \ge 0 \\ 0 & t < 0. \end{cases}$$

Observe that $P_x(\tau_1 \ge t) = \int_t^\infty q_x e^{-q_x s}\, ds = e^{-q_x t}$, $t \ge 0$

If x is an absorbing state, we set $q_x = 0$.

It follows from the Markov property that for $0 \le t_1 \le \cdots \le t_n$ and $x_1, x_2, \ldots, x_n \in \mathcal{S}$,

$$P(X(t_1) = x_1, \ldots, X(t_n) = x_n)$$
$$= P(X(t_1) = x_1)\, P_{x_1 x_2}(t_2 - t_1) \cdots P_{x_{n-1} x_n}(t_n - t_{n-1})$$

In particular, for $s \ge 0$ and $t \ge 0$

$$P_x(X(t) = z, X(t+s) = y) = P_{xz}(t)\, P_{zy}(s)$$

Since $P_{xy}(t+s) = \sum_z P_x(X(t) = z, X(t+s) = y)$

we conclude that

$$P_{xy}(t+s) = \sum_z P_{xz}(t)\, P_{zy}(s), \quad s \ge 0, t \ge 0$$

This is known as the Chapman-Kolmogorov equation

Birth and death process

Let $\mathcal{S} = \{0, 1, \ldots d\}$ or $\mathcal{S} = \{0, 1, 2, \ldots\}$.

By a birth and death process on $\mathcal{S}$ we mean a Markov pure jump process on $\mathcal{S}$ having infinitesimal parameters q_{xy} such that

$$q_{xy} = 0, \quad |y - x| > 1$$

Thus a birth and death process stating at x can in one jump go only to the states $x-1$ or $x+1$ and The parameters $\lambda_x = q_{x,x+1}$, $x \in \mathcal{S}$ and $\mu_x = q_{x,x-1}$, $x \in \mathcal{S}$ are called respectively the birth rates and death rates of the process. The parameters q_x and Q_{xy} of the process can be expressed simply in terms of the birth and death rates.

$$\text{Now, } -q_{xx} = q_x = q_{x,x+1} + q_{x,x-1} \quad ?$$

So that $q_{xx} = -(\lambda_x + \mu_x)$ and $q_x = \lambda_x + \mu_x$.

Thus, x is an absorbing state iff $\lambda_x = \mu_x = 0$.

If x is a non-absorbing state, then

$$Q_{xy} = \begin{cases} \dfrac{\mu_x}{\lambda_x + \mu_x} & y = x-1 \\[2ex] \dfrac{\lambda_x}{\lambda_x + \mu_x} & y = x+1 \\[2ex] 0 & \text{elsewhere} \end{cases}$$

A birth and death process is called a pure birth process if $\mu_x = 0$, $x \in \mathcal{S}$, and a pure death if $\lambda_x = 0$, $x \in \mathcal{S}$. A pure birth process can move only to the right, and a pure death process can move only to the left.

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Branching Process

Consider a collection of particles which act ~~indep~~ independently in giving rise to succeeding generations of particles. Suppose that each particle, from the time it appears, waits a random length of time having an exponential distribution with parameter q and then splits into two identical particles with ~~propor~~ probability p and disappears with probability $1-p$. Let $X(t)$, $0 \leq t < \infty$, denotes the number of particles present at time t. This branching process is a birth and death process.

Consider a brancing process starting out with x particles. Let $\xi_1, \xi_2, \ldots, \xi_x$ be the time until these particles split apart or disappear. Then $\xi_1, \xi_2, \ldots, \xi_x$ each has an exponential distribution with parameter q, and hence $\tau_1 = \min(\xi_1, \xi_2, \ldots \xi_x)$ has an exponential distribution with parameter $q_x = xq$. Whichever particle acts first has probability p of spliting into two particles and probability $1-p$ of disappearing. Thus for $x \geq 1$

$$Q_{x,x+1} = p. \quad \text{and} \quad Q_{x,x-1} = 1-p.$$

state 0 is an absorbing state.

Since, $\lambda_x = q_x Q_{x,x+1}$ and $\mu_x = q_x Q_{x,x-1}$,
we conclude that $\lambda_x = xpq$ and $\mu_x = xq(1-p)$

This processs is a bioth and death process. This intuitively reasonable property basically depends on the fact that an exponentially distributed random variable ξ satisfies the formula

$$P(\xi > t+s \mid \xi > s) = P(\xi > t) \qquad s,t \geq 0.$$

~~bad~~

Since $q_{xx} = -(\lambda_x + \mu_x)$ and $q_x = \lambda_x + \mu_x$

the backward and forward equation of this process are

$$P'_{xy}(t) = \mu_x P_{x-1,y}(t) - (\lambda_x + \mu_x) P_{xy}(t) + \lambda_x P_{x+1,y}(t), \quad t \geq 0$$

and

$$P'_{xy}(t) = \lambda_{y-1} P_{x,y-1}(t) - (\lambda_y + \mu_y) P_{xy}(t) + \mu_{y+1} P_{x,y+1}(t), \quad t \geq 0$$

Here we set $\lambda_{-1} = 0$ and if $\mathcal{S} = \{0, 1, \ldots d\}$ for $d < \infty$, we set $\mu_{d+1} = 0$.

Branching process with immigration

Consider the branching process introduced in previous example. Suppose that new particles immigrate into the system at random times that form a Poisson process with parameter λ and then give rise to succeeding generations as described previously.

Suppose there are initially x particles present. Let $\xi_1, \xi_2, \ldots, \xi_x$ be the times at which these particles split apart or disappear, and let η be the first time a new particle enters in the system. We ~~imper~~ interpret the description of the system as implying that η is independent of $\xi_1, \xi_2, \ldots, \xi_x$. Then $\xi_1, \xi_2, \ldots \xi_x$ and η are independent exponentially distributed random variables having respective parameters $q, \ldots, q, \lambda$.

Thus $\tau_1 = \min\{\xi_1, \xi_2, \ldots, \xi_x, \eta\}$

is exponentially distributed with parameter

$q_x = xq + \lambda$ and then

$$P(\tau_1 = \eta) = \frac{\lambda}{xq + \lambda}.$$

The event $\{X(\tau_1) = x+1\}$ occurs if either $\tau_1 = \eta$

or $\tau_1 = \min\{\xi_1, \ldots \xi_x\}$

and a particle splits into two new particles at time τ_1.

thus. $$Q_{x,x+1} = \frac{\lambda}{xq+\lambda} + \frac{xq}{xq+\lambda} p.$$

Also, $$Q_{x,x-1} = \frac{xq}{xq+\lambda} (1-p)$$

$\therefore \lambda_x = q_x Q_{x,x+1} = xpq + \lambda$ (check)

and $\mu_x = q_x Q_{x,x-1} = xq(1-p)$ (check)

It is also possible to construct a Poisson process with parameter λ on $-\infty < t < \infty$. We first construct two independent Poisson processes $X_1(t)$, $0 \le t < \infty$, and $X_2(t)$, $0 \le t < \infty$, both having parameter λ.

We then define $X(t)$, $-\infty < t < \infty$ by

$$X(t) = \begin{cases} -X_1(-t) & t < 0, \\ X_2(t) & t \geq 0. \end{cases}$$

It is easy to show that the process $X(t)$, $-\infty < t < \infty$ satisfies the following properties:

i) $X(0) = 0$

ii) $X(t) - X(s)$ has a Poisson distribution with parameter $\lambda(t-s)$ for $s \le t$,

iii) $X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are independent for $t_1 \le t_2 \le \cdots \le t_n$.

Branching process with immigration

Consider the branching process introduced in previous example. Suppose that new particles immigrate into the system at random times that form a Poisson process with parameter λ and then give rise to succeeding generations as described previously.

Suppose there are initially x particles present. Let $\xi_1, \xi_2, \ldots, \xi_x$ be the times at which these particles split apart or disappear, and let η be the first time a new particle enters in the system. We ~~imper~~ interpret the description of the system as implying that η is independent of $\xi_1, \xi_2, \ldots, \xi_x$. Then $\xi_1, \xi_2, \ldots \xi_x$ and η are independent exponentially distributed random variables having respective parameters $q, \ldots, q, \lambda$.

Thus $\tau_1 = \min\{\xi_1, \xi_2, \ldots, \xi_x, \eta\}$ is exponentially distributed with parameter

$q_x = xq + \lambda$ and then

$$P(\tau_1 = \eta) = \frac{\lambda}{xq + \lambda}.$$

The event $\{X(\tau_1) = x+1\}$ occurs if either $\tau_1 = \eta$ or $\tau_1 = \min\{\xi_1, \ldots \xi_x\}$ and a particle splits into two new particles at time τ_1.

$$Q_{x,x+1} = \frac{\lambda}{xq+\lambda} + \frac{xq}{xq+\lambda}\, p.$$

Also, $$Q_{x,x-1} = \frac{xq}{xq+\lambda}\,(1-p)$$

$\therefore\ \lambda_x = q_x Q_{x,x+1} = xpq + \lambda$ (check)

and $\mu_x = q_x Q_{x,x-1} = xq(1-p)$ (check)

It is also possible to construct a Poisson process with parameter λ on $-\infty < t < \infty$. We first construct two independent Poisson processes $X_1(t)$, $0 \le t < \infty$, and $X_2(t)$, $0 \le t < \infty$, both having parameter λ.

We then define $X(t)$, $-\infty < t < \infty$ by

$$X(t) = \begin{cases} -X_1(-t) & t < 0, \\ X_2(t) & t \geq 0. \end{cases}$$

It is easy to show that the process $X(t)$, $-\infty < t < \infty$ satisfies the following properties:

i) $X(0) = 0$

ii) $X(t) - X(s)$ has a Poisson distribution with parameter $\lambda(t-s)$ for $s \le t$,

iii) $X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are independent for $t_1 \le t_2 \le \ldots \le t_n$.