6. Triangulations

6.1 Triangulating spaces

The collection of all topological spaces is much too vast for us to work with. We have seen in previous chapters how to develop an abstract theory of topological spaces and continuous functions and to prove many important results. However, working in such a general setting we quickly run into two kinds of difficulty. On the one hand, in trying to prove a concrete geometrical result such as the classification theorem for surfaces, the purely topological structure of the surface (that it be locally euclidean) does not give us much leverage from which to start. On the other hand, although we can define algebraic invariants, such as the fundamental group, for topological spaces in general, they are not a great deal of use to us unless we can *calculate* them for a reasonably large collection of spaces. Both of these problems may be dealt with effectively by working with spaces that can be broken up into pieces which we can recognize, and which fit together nicely, the so called *triangulable* spaces.

Fig. 6.1 shows the sort of construction we have in mind. A homeomorphism from the surface of a tetrahedron to the sphere gives a decomposition of the sphere into four triangles, the triangles being joined along their edges. As a

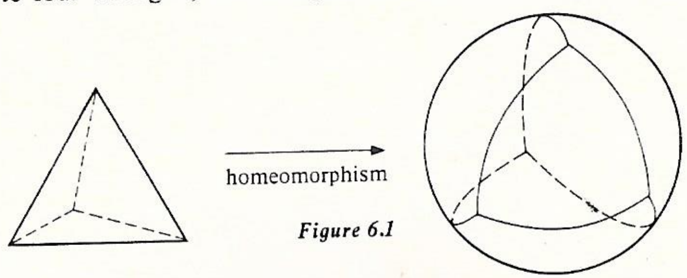


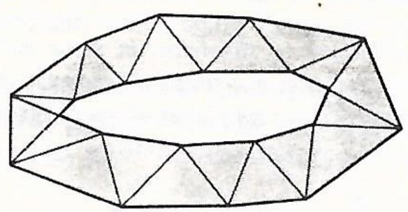
Figure 6.1

second example, suppose we chop up a strip into triangles and then identify its ends with a half twist (Fig. 6.2). We obtain a space homeomorphic to a Möbius strip and we say that we have 'triangulated' the Möbius strip.

Both the sphere and the Möbius strip are surfaces. They are two-dimensional and so we can make models of them using triangles. For spaces of higher dimension we need higher-dimensional building blocks for our construction.

Let $v_0, v_1, \ldots, v_k$ be points of euclidean n-space $\mathbb{E}^n$. The hyperplane spanned by these points consists of all linear combinations $\lambda_0 v_0 + \lambda_1 v_1 + \ldots + \lambda_k v_k$, where each λ_i is a real number and the sum of the λ_i is 1. The points are in *general*

position if any subset of them spans a strictly smaller hyperplane. It is an easy matter to check that if we regard $\mathbb{E}^n$ as a vector space, then this is equivalent to asking that the vectors $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_k - \mathbf{v}_0$ be linearly independent.



A simplicial complex homeomorphic to the Möbius strip

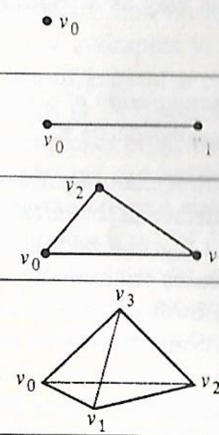
Figure 6.2

Given $k + 1$ points $v_0, v_1, \ldots, v_k$ in general position, we call the smallest convex set containing them a *simplex of dimension* k (or a *k-simplex*). The points $v_0, v_1, \ldots, v_k$ are called the *vertices* of the simplex. We recall that a point x lies in the smallest convex set containing $v_0, v_1, \ldots, v_k$ if and only if it can be written as a linear combination

$$x = \lambda_0 v_0 + \lambda_1 v_1 + \ldots + \lambda_k v_k$$

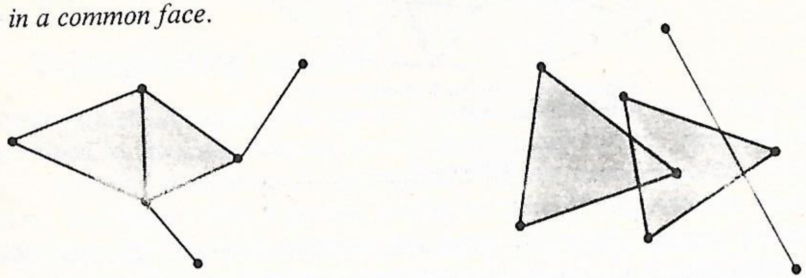
where the λ_i are all nonnegative real numbers and $\lambda_0 + \lambda_1 + \ldots + \lambda_k = 1$. Looking at the first few dimensions we obtain:

0-simplex = point	v_0
1-simplex = closed line segment	v_0, v_1
2-simplex = triangle	v_0, v_1, v_2
3-simplex = tetrahedron (solid)	v_0, v_1, v_2, v_3



Simplexes have 'faces' in a natural way. If A and B are simplexes and if the vertices of B form a subset of the vertices of A, then we say that B is a *face* of A and write $B < A$. The idea of simplexes fitting together 'in a nice way' can be made precise by asking that if two simplexes intersect, then they do so in a common face (Fig. 6.3). We shall call a space *triangulable* if it is homeomorphic to the union of a finite collection of simplexes which fit together nicely in some euclidean space. We now look into this idea in a little more detail.

(6.1) Definition. *A finite collection of simplexes in some euclidean space* $\mathbb{E}^n$ *is called a simplicial complex if whenever a simplex lies in the collection then so does each of its faces, and whenever two simplexes of the collection intersect they do so in a common face.*



Simplexes which fit together nicely

The sort of intersections that are *not* allowed

Figure 6.3

We shall use letters such as K, L for simplicial complexes, reserving X and Y to denote topological spaces. Now the union of the simplexes which make up a particular complex† is a subset of a euclidean space, and can therefore be made into a topological space by giving it the subspace topology. A complex K, made into a topological space by giving it the subspace topology, when regarded in this way as a topological space, is called a *polyhedron* and written $|K|$.

(6.2) Definition. *A triangulation of a topological space* X *consists of a simplicial complex* K *and a homeomorphism* h : $|K| \to X$.

Going back to our first example, X is the sphere, K the collection of simplexes which make up the surface of the tetrahedron, and, if the tetrahedron lies inside the sphere as in Fig. 1.8, h can be taken to be radial projection.

Asking that a space be triangulable is of course asking a great deal. A simplicial complex K is built up of a finite number of simplexes which live in a euclidean space, and consequently its polyhedron $|K|$ will have many pleasant properties: for example, it will be compact and a metric space. Therefore if a space is to be triangulable it must possess these properties. None the less, many important spaces admit a triangulation; in Chapter 7 we shall make essential use of the fact that all closed surfaces are triangulable.

† We often omit the word simplicial.

Triangulations are not unique.† The definition of a triangulation leaves us a great deal of choice, namely the choice of the simplicial complex K and of the triangulating homeomorphism h. A triangulation should be regarded as a *tool* which helps us to prove a particular result or do some calculation. It is its existence that is important: which triangulation we use is often of no great relevance.

A model for a triangulation of the torus is shown in Fig. 6.4. Making the identifications indicated via arrows on the edges of the rectangle, one can build a simplicial complex in $\mathbb{E}^3$ whose polyhedron is homeomorphic to the torus.

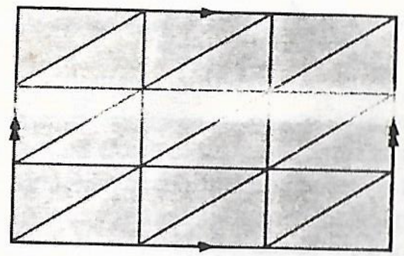


Figure 6.4

By definition, a simplicial complex always consists of simplexes which lie in some euclidean space $\mathbb{E}^n$. If we wish to emphasize the role played by the euclidean space, we say that K is a complex in $\mathbb{E}^n$. (We emphasize that K is a collection of simplexes and not a set of points.) Regard $\mathbb{E}^n$ as the subspace of $\mathbb{E}^{n+1}$ consisting of those points which have final coordinate zero. We can construct a complex CK in $\mathbb{E}^{n+1}$, which is called *the cone on* K, as follows. Let v denote the point $(0,0,\ldots,0,1)$ in $\mathbb{E}^{n+1}$. If A is a k-simplex in $\mathbb{E}^n$ with vertices $v_0, v_1, \ldots, v_k$, then the points $v_0, v_1, \ldots, v_k, v$ are in general position and therefore determine a $(k+1)$-simplex in $\mathbb{E}^{n+1}$. This $(k+1)$-simplex is called the *join* of A to v. Our cone CK consists of the simplexes of K, the join of each of these simplexes to v, and the 0-simplex v itself. One can easily check that the simplexes of this collection do fit together nicely and form a simplicial complex. CK is often called the join of K to v. As a set of points in $\mathbb{E}^{n+1}$, its polyhedron consists of all straight-line segments joining v to some point of $|K|$ (Fig. 6.5). In Chapter 4 we defined the cone CX on an arbitrary topological space X. The two ideas coincide in the sense that $|CK|$ and $C|K|$ are homeomorphic topological spaces (see lemma 4.5).

This cone construction gives us an easy way of triangulating the projective plane P. Recall that P is formed by taking a Möbius strip and a disc and sewing their boundaries together. Now we have already triangulated the Möbius strip M by means of the simplicial complex K in $\mathbb{E}^3$ shown in Fig. 6.2. Let L

† The only space with a unique triangulation consists of a single point.

consist of those simplexes of K which triangulate the boundary of M, i.e., the nineteen 1-simplexes and nineteen vertices which in our picture form the edge of K. Then $K \cup CL$ is a complex in $\mathbb{E}^4$ whose polyhedron is homeomorphic to the projective plane, for $|K|$ is homeomorphic to M and $|CL|$ is, up to homeomorphism, just a cone with base a circle, i.e., a disc. L as defined above is an

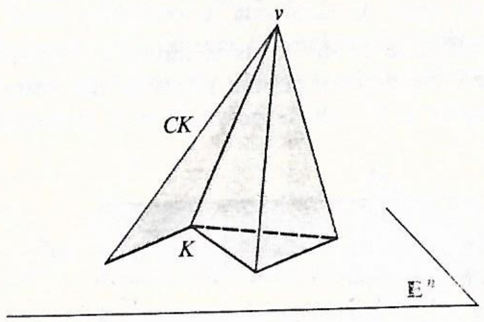


Figure 6.5

example of a *subcomplex* of a simplicial complex, i.e., it is a subcollection of the simplexes of a complex K which itself forms a complex.

In defining the cone on a complex K, we had to make a choice of point to represent the apex of the cone. We chose a point outside $\mathbb{E}^n$ to ensure that adding this new point to the set of vertices of a simplex of K produced a set of points in general position. But why choose v; why not choose some other point of $\mathbb{E}^{n+1} - \mathbb{E}^n$? A different choice would give a different set of simplexes in $\mathbb{E}^{n+1}$, but the simplexes would intersect one another in the same sort of way as the simplexes of CK. This leads us naturally to the idea of two simplicial complexes being isomorphic. Let K and L be complexes, not necessarily in the same euclidean space. They are *isomorphic* if there is a bijection ϕ from the set of vertices of K to the set of vertices of L such that $v_1, v_2, \ldots, v_s$ form the vertices of a simplex of K if and only if $\phi v_1, \phi v_2, \ldots, \phi v_s$ form the vertices of a simplex of L. The notion of isomorphism has nothing to do with the particular euclidean spaces in which the complexes lie, or the way in which their polyhedra are embedded in these euclidean spaces. It is simply a statement that K and L have the same number of simplexes of each dimension *and* that these simplexes exhibit the same pattern of intersections. The most important thing about isomorphic complexes is that they have *homeomorphic polyhedra.* Try to prove this. (The function ϕ is defined only on the vertices of K; try to extend it 'linearly' over each simplex of K to construct a homeomorphism from $|K|$ to $|L|$. We shall give the details of this construction in Section 6.3.) Now if $v, w \in \mathbb{E}^{n+1} - \mathbb{E}^n$, then the join of K to v and the join of K to w are isomorphic complexes (use the identity function on the vertices of K and send v to w). So our choice of apex in $\mathbb{E}^{n+1} - \mathbb{E}^n$ does not really matter.

We close this section by noting, for future reference, one or two facts concerning simplicial complexes. Let A be a simplex in $\mathbb{E}^n$ with vertices $v_0, v_1, \ldots, v_k$.

We define the *interior* of A to consist of those points x of A which can be written in the form $x = \lambda_0 v_0 + \lambda_1 v_1 + \ldots + \lambda_k v_k$ where $\sum_0^k \lambda_i = 1$ and the λ_i are *all positive.* Note that this notion coincides with the topological definition of interior when $k = n$, but not otherwise.

(6.3) Lemma. *Let* K *be a simplicial complex in* $\mathbb{E}^n$.
(a) $|K|$ *is a closed bounded subset of* $\mathbb{E}^n$, *and so* $|K|$ *is a compact space.*
(b) *Each point of* $|K|$ *lies in the interior of exactly one simplex of* K.
(c) *If we take the simplexes of* K *separately and give their union the identification topology, then we obtain exactly* $|K|$.
(d) *If* $|K|$ *is a connected space, then it is path-connected.*

Proof. Each simplex of K is closed and bounded. Since K is finite, the result (a) follows. For (b), suppose A and B are simplexes of K whose interiors overlap. Since K is a complex, A and B are required to meet in a common face. But the only face of a simplex which contains interior points is the whole simplex itself. Therefore $A = B$. In (c) we note that simplexes of K are closed subsets of $|K|$ since they are closed in $\mathbb{E}^n$. So if C is a subset of $|K|$, and if $C \cap A$ is closed in A for each simplex A of K, then $C \cap A$ must be closed in $|K|$. Therefore the *finite* union $C = \bigcup \{C \cap A \mid A \in K\}$ is closed in $|K|$. So the closed subsets of $|K|$ are precisely those which intersect each simplex of K in a closed set, in other words $|K|$ has the identification topology. Finally, for part (d), suppose $|K|$ is connected. Given $x \in |K|$, let L denote the subcomplex of K consisting of all those simplexes of K that do not contain x, and let ε denote the distance from x to $|L|$. Then if $\delta < \varepsilon$ the set $B(x, \delta) \cap |K|$ is path-connected, because any point in this set can be joined to x by a straight line in some simplex of K. This means that $|K|$ is a locally path-connected space, and we can mimic the proof of theorem (3.30) to show that it is path-connected.

Problems

1. Construct triangulations for the cylinder, the Klein bottle, and the double torus.

2. Finish off the proof of lemma (6.3).

3. If $|K|$ is a connected space, show that any two vertices of K can be connected by a path whose image is a collection of vertices and edges of K.

4. Check that $|CK|$ and $C|K|$ are homeomorphic spaces.

5. If X and Y are triangulable spaces, show that $X \times Y$ is triangulable.

6. If K and L are complexes in $\mathbb{E}^n$, show that $|K| \cap |L|$ is a polyhedron.

7. Show that S^n and P^n are both triangulable.

8. Show that the 'dunce hat' (Fig. 5.11) is triangulable, but that the 'comb space' (Fig. 5.10) is not.

6.2 Barycentric subdivision

Let K be a simplicial complex in $\mathbb{E}^n$. In this section we describe a construction which allows us to chop up the simplexes of K and produce a new complex K^1, which has the *same polyhedron* as K, but which has *simplexes of smaller diameter*.

The process is called 'barycentric subdivision'. If A is a simplex of K with vertices $v_0,v_1,\ldots,v_k$, then each point x of A has a unique expression of the form $x = \lambda_0 v_0 + \lambda_1 v_1 + \ldots + \lambda_k v_k$ where $\sum_0^k \lambda_i = 1$ and all the λ_i are nonnegative. These numbers λ_i are called the *barycentric coordinates* of the point x, and the *barycentre* (or centre of gravity) of A is the point

$$\hat{A} = \frac{1}{k+1}(v_0 + v_1 + \ldots + v_k).$$

In order to form K^1 we begin by adding extra vertices to K at the barycentres of its simplexes. Then, working in order of increasing dimension, we chop up each simplex of K as a cone with apex the extra vertex at its barycentre. Figure 6.6 illustrates the process.

To define K^1 precisely, we need to describe its simplexes. The vertices of K^1 are the barycentres of the simplexes of K. (This includes the original vertices of K since a 0-simplex is its own barycentre.)

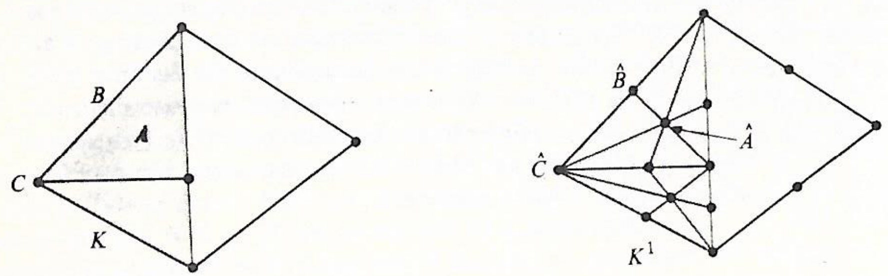


Figure 6.6

A collection $\hat{A}_0, \hat{A}_1,\ldots,\hat{A}_k$ of such barycentres form the vertices of a k-simplex of K^1 if and only if

$$A_{\sigma(0)} < A_{\sigma(1)} < \ldots < A_{\sigma(k)}$$

for some permutation σ of the integers $0,1,2,\ldots,k$. For example, in our illustration the barycentres $\hat{A}, \hat{B}, \hat{C}$ determine a 2-simplex of K^1, and looking at K we see $C < B < A$. Note that if $A_{\sigma(0)} < A_{\sigma(1)} < \ldots < A_{\sigma(k)}$, then for each i the barycentre $\hat{A}_{\sigma(i)}$ lies off the hyperplane spanned by $\hat{A}_{\sigma(0)},\ldots,\hat{A}_{\sigma(i-1)}$. Consequently, the points $\hat{A}_{\sigma(0)},\ldots,\hat{A}_{\sigma(k)}$ are in general position.

The *dimension* of a simplicial complex K is the maximum of the dimensions of its simplexes, and its *mesh* $\mu(K)$ is the maximum of the diameters of its simplexes.

(6.4) Lemma. *The collection of simplexes described above forms a simplicial complex. It is denoted by* K^1 *and is called the first barycentric subdivision of* K. K^1 *has the following properties:*

(a) *each simplex of* K^1 *is contained in a simplex of* K;

(b) $|K^1| = |K|$;

(c) *if the dimension of* K *is* n, *then* $\mu(K^1) \leqslant \frac{n}{n+1}\mu(K)$.

Proof. If σ is a simplex of K^1, we can label its vertices $\hat{A}_0,\hat{A}_1,\ldots,\hat{A}_k$, where the A_i belong to K and $A_0 < A_1 < \ldots < A_k$. So all the vertices of σ lie in A_k, and therefore σ is contained in A_k. This proves property (a). Note that any face of σ lies in K^1, so in checking that K^1 is a simplicial complex we need only verify that its simplexes fit together nicely.

We shall prove that K^1 is a complex and satisfies $|K^1| = |K|$ by induction on the number of simplexes of K. The induction begins trivially when K consists of a single vertex. Suppose the result is true for all complexes which have less than m simplexes, and let K be a complex which is made up of m simplexes. Choose a simplex A of maximum dimension in K, and form a new complex L by removing A from K. Then L has $m-1$ simplexes and its polyhedron consists of $|K|$ with the interior of the simplex A deleted. By the inductive hypothesis, L^1 is a simplicial complex and $|L^1| = |L|$. We need to look at the simplexes of K^1 that do not lie in L^1. Let σ be such a simplex (σ not equal to $\hat{A}$) and label its vertices as $\hat{A}_0,\hat{A}_1,\ldots,\hat{A}_{k-1}, \hat{A}$ where $A_0 < A_1 < \ldots < A_{k-1} < A$. The vertices $\hat{A}_0,\hat{A}_1,\ldots,\hat{A}_{k-1}$ determine a face τ of σ which lies in L^1, and $\tau = \sigma \cap |L^1|$. Therefore if σ meets a simplex of L^1, it must do so in a face of τ, and consequently in one of its own faces. Let σ' be a second simplex of $K^1 - L^1$ (again, not the vertex $\hat{A}$) and define τ' as above. Then if τ and τ' intersect, they do so in a common face (since L^1 is a complex). In this case the vertices of $\tau \cap \tau'$ together with $\hat{A}$ determine a common face of σ and σ' which is exactly $\sigma \cap \sigma'$. If τ and τ' do not intersect, then σ and σ' intersect in the vertex $\hat{A}$. Therefore K^1 is a simplicial complex.

Each simplex of K^1 being contained in a simplex of K, we know that $|K^1| \subseteq |K|$; so we now prove the reverse inclusion. Let $x \in |K|$ and let A be the unique simplex of K which contains x in its interior. If $x = \hat{A}$, then certainly $x \in |K^1|$. If not, join $\hat{A}$ to x by a straight line and prolong the line until it meets a face of A. Call the intersection point y. Then $y \in |L| = |L^1|$, and so $y \in \tau$ for some simplex τ of L^1. The vertices of τ together with $\hat{A}$ determine a simplex of K^1 which contains x. Therefore $x \in |K^1|$ and we have proved $|K^1| = |K|$, which is property (b).

It remains to verify property (c). First observe that the diameter of a simplex is the length of its longest edge. Let σ be an edge of K^1 with vertices $\hat{A}$ and $\hat{B}$, say, where $B < A$. Then σ is contained in A, and if the dimension of A is k we have

$$\text{length } \sigma \leqslant \frac{k}{k+1}(\text{diameter } A) \leqslant \frac{n}{n+1}(\text{diameter } A) \leqslant \frac{n}{n+1}\mu(K)$$

Therefore $\mu(K^1) \leqslant \frac{n}{n+1}\mu(K)$.

Define the *m-th barycentric subdivision* K^m of K inductively by $K^m = (K^{m-1})^1$. Figure 6.7 shows K^2 when K consists of a 2-simplex plus all its faces. Property (c) of Lemma (6.4) tells us that, by taking m large enough, we can make the diameters of the simplexes of K^m as small as we like.

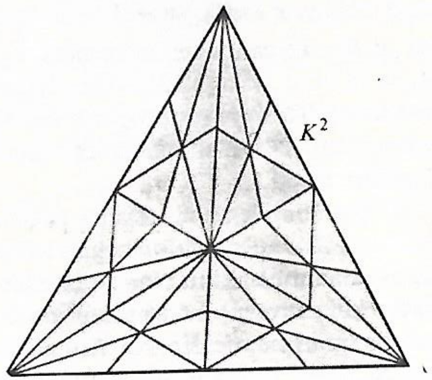


Figure 6.7

Problems

9. Make sure you can visualize the first barycentric subdivision of a 3-simplex.

10. Let $\mathscr{F}$ be an open cover of $|K|$. Show the existence of a barycentric subdivision K^r with the property that given a vertex v of K^r, there is an open set U in $\mathscr{F}$ which contains all the simplexes of K^r that have v as a vertex.

11. Let L be a subcomplex of K, and let N be the following collection of simplexes of K^2: a simplex B lies in N if we can find a simplex C in L^2 such that the vertices of B and C together determine a simplex of K^2. Show that N is a subcomplex of K^2, and that $|N|$ is a neighbourhood of $|L|$ in $|K|$.

12. Use the construction of Problem 11 to prove that if X is a triangulable space, and Y a subspace of X which is triangulated by a subcomplex of some triangulation of X, then the space obtained from X by shrinking Y to a point is triangulable.

6.3 Simplicial approximation

Let X and Y be topological spaces with triangulations $h:|K| \to X$, $k:|L| \to Y$. Then any map $f:X \to Y$ automatically induces a map $k^{-1}fh:|K| \to |L|$. There is a particular kind of map between polyhedra which is easy to work with, namely the so-called simplicial map which takes simplexes to simplexes, and which is *linear* on each simplex. In many problems, for example in calculating

the fundamental group of a triangulable space, it is important to be able to approximate a given map by a simplicial map. The approximation we choose will be close enough to the given map so that the two are homotopic; i.e., the approximation can be continuously deformed into the original map.

(6.5) Definition. *Let* K *and* L *be simplicial complexes. A function* $s:|K| \to |L|$ *is called simplicial if it takes simplexes of* K *linearly onto simplexes of* L.

Writing this out in detail: if A is a simplex of K, we require $s(A)$ to be a simplex of L; the condition of linearity means that if A has vertices $v_0, v_1, \ldots, v_k$, and if $x \in A$ is the point $x = \lambda_0 v_0 + \lambda_1 v_1 + \ldots + \lambda_k v_k$, where the λ_i are nonnegative and $\sum_0^k \lambda_i = 1$, then $s(x)$ when expressed in terms of the vertices of $s(A)$ is $s(x) = \lambda_0 s(v_0) + \lambda_1 s(v_1) + \ldots + \lambda_k s(v_k)$. Note that $s(A)$ may have lower dimension than A (we do not require s to be one–one), in which case $s(v_0),\ldots,s(v_k)$ will not all be distinct.

It should be clear that a simplicial function is *continuous*. This follows from the fact that a linear function between two simplexes is continuous, and application of the glueing lemma (4.6).

Because of its linearity on each simplex of K, a simplicial map s is completely determined once we know its effect on the vertices of K. In fact, if a function s from the vertices of K to the vertices of L has the property that if vertices $v_0, v_1, \ldots, v_k$ determine a simplex of K then $s(v_0), \ldots, s(v_k)$ determine a simplex of L, then s can be extended linearly across each simplex of K to give a simplicial map $|K| \to |L|$. In particular, an isomorphism from K to L extends in this way to a simplicial homeomorphism from the polyhedron of K to the polyhedron of L.

Now let $f:|K| \to |L|$ be a map between polyhedra. Given a point $x \in |K|$, the point $f(x)$ lies in the interior of a unique simplex of L. Call this simplex the *carrier* of $f(x)$.

(6.6) Definition. *A simplicial map* $s:|K| \to |L|$ *is a simplicial approximation of* $f:|K| \to |L|$ *if* $s(x)$ *lies in the carrier of* $f(x)$ *for each* $x \in |K|$.

Note that if s simplicially approximates f, then s *and* f *are homotopic*. This follows immediately from the definition. For suppose L lies in $\mathbb{E}^n$, and let $F:|K| \times I \to \mathbb{E}^n$ denote the straight-line homotopy defined by $F(x,t) = (1-t)s(x) + tf(x)$. Given $x \in |K|$, we know that some simplex of L contains $s(x)$ and $f(x)$ and, since a simplex is convex, all points $(1-t)s(x) + tf(x)$, $0 \leqslant t \leqslant 1$, must also lie in this simplex. Therefore the image of F lies in $|L|$, and F is a homotopy from s to f.

Simplicial approximations do not always exist (see Example (6.8) below). However, we can guarantee their existence if we are prepared to replace K by a suitable barycentric subdivision K^m.

(6.7) Simplicial approximation theorem. *Let* $f:|K| \to |L|$ *be a map between*